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ELEMENTS

OF

ALGEBRA;

TO WHICH IS PREFIXED,

A CHOICE COLLECTION

OF

ARITHMETICAL QUESTIONS,
WITH THEIR SOLUTIONS,

INCLUDING

SOME NEW IMPROVEMENTS WORTHY THE
ATTENTION OF ARITHMETICIANS.

The PRINCIPLES of ALGEBRA are clearly demonſtrated, and applied in the Reſolution of a great Variety of PROBLEMS on different Parts of the

MATHEMATICKS AND NATURAL PHILOSOPHY.

By JOHN MOLE,
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LONDON:
Printed for G. G. J. and J. ROBINSON, Paternoſter-Row,
MDCCLXXXVIII.

To the READER.

AN EULOGY ON ALGEBRA:

A TREATISE on this ſubject might be conſidered almoſt ſuperfluous, were not the excellence of Algebra in itſelf as a ſcience, and its extenſive utility in thoſe to which it may be applied, now pretty generally known: Nor is it much more neceſſary to apologize for the preſent Publication from the number already extant on the ſame ſubject. The improvement of knowledge is gradual, and ariſes from repeated accumulations: the moſt trivial inſtruments which may contribute to ſo deſirable an end, ought not to be unemployed; and whoever thinks his obſervations on any ſcience capable of extending its bounds, or of removing thoſe obſtacles, which may have retarded the progreſs of learners in tracks

already

already explored, is justified in communicating them to the public.

It may be proper, however, to acquaint the reader with the end which this Treatise is intended to answer, and the method that has been observed in composing it: it is designed then merely as an introduction to the science, and to render it attainable without the assistance of a teacher. The elegance of conciseness, has been therefore frequently made to give place to perspicuity, and a care to be understood has ever been the first in view. Some excellent pieces, it is confessed, have been published professedly on the same plan, but many things still remain in the writings of the higher class of authors, which often frustrate the success of unassisted endeavours: whether this arises from real obscurity in those writings, or from any other cause, we do not presume to determine; there may perhaps be a necessity of presenting truth in various lights, to render it intelligible to different capacities.

It must be allowed, that some writers on Algebra have done more in large volumes than can be expected here; but, however, on perusing this through, it will be found to comprise the most essential principles of the science, and as comprehensive as its plan and limits would permit: the author has not unnecessarily swelled the work, in solving Problems by formal Solutions of such Equations as he had plainly treated of before; neither

has

has he endeavoured to gain a reputation for profoundneſs, by leaving a great deal to be gueſſed at by the reader; well knowing by experience, that a ſmall matter turns a perſon untutored quite out of his way; and it muſt be granted, that a work may be too hard for ſome, but it never can be too eaſy for any: ſuffice it then to ſay, that the author has ſpared no pains to make this as intelligible, in his humble opinion, as the nature of the ſubject will admit.

Algebra, as a method of reaſoning, may be employed to inveſtigate inquiries in various branches of knowledge; it is an art truly ſublime, and of unlimited extent, but it is to be obſerved, that our reaſonings muſt ever be founded on ſome evident or known principles; and as mathematical ſtudies ſeldom commence with Algebra, ſome previous qualifications are ſuppoſed to be attained, and it is preſumed here, that the reader is well verſed in Arithmetic, and in the uſe of Logarithms; that he underſtands the Rudiments of Geometry and of Trigonometry both plain and ſpherical.

To the Algebra is prefixed a ſelect collection of queſtions relating to numbers, and to the ſolutions of ſuch of them as do not come within the rules of common Arithmetic, explanatory notes are inſerted, in order to excite and encourage young arithmeticians.

The Algebra begins with Notation, and is continued on gradually with Addition, Subtraction,

traction, Multiplication, Division, Reduction, &c. &c. to the Solution of simple Equations: Then after other necessary preliminaries; and various Problems, you have the Resolution of Quadratic Equations.

Among the Problems producing Quadratics, are derived many Equations which are really affected Biquadratics, but by connecting their terms, they are here solved by Quadratics: These are intended here, to familiarize and initiate the learner in that which he will find amply treated of further on.

To the general Solution of Cubic Equations are added some remarks, showing how and when a Cubic Equation can be solved by compleating the Cube.

Next follows the Resolution of affected Biquadratics, these are very largely expatiated on, copiously interspersed in the course of the work, and solved by methods which are as easy as they are new.

The Author's design in expatiating so largely on these Equations, is to enable the learner, and those not well acquainted with this subject, to determine, when an affected Biquadratic can be solved by a Quadratic. This is a circumstance of great importance, and the author flatters himself that it is not very generally understood.

In Converging Series, he has invented a method by which the Roots of Biquadratic

and higher Equations, may be readily obtained to a very great degree of exactneſs.

Among the Exponential Equations, you have a new and eaſy method of Solution, when the given numbers are too big for the courſe of the Logarithmic Tables.

The Author has likewiſe introduced a method of ſubſtituting for the variable Exponents, by which the roots of two or more Equations containing as many unknown quantities, may be determined to any aſſigned degree of accuracy.

This method commences with Example VI. on page 236.

It would be needleſs to enumerate all the particulars here, ſince there is a Table of Contents, to which we refer the reader for a further account of the work.

We ſhall only obſerve, that the principles are laid down in the cleareſt manner, and exercifed in the Solution of a variety of Problems, many of which have before been publiſhed; but the operations will be here found leſs intricate to trace, as they are ſolved by the moſt ſimple Equations poſſible: ſome are new, and theſe it is hoped, will be found, both from their nature, and the manner of ſolution, adapted to improve and amuſe the learner.

Every work will meet with ſucceſs in the world nearly proportionate to its intrinſic merit; it would be of little avail therefore, to allege the diſadvantages under which the preſent

ſent was compoſed, as a palliative for its imperfections. Thoſe who know the ſituation of the writer, know alſo the degree of indulgence to which he is intitled; but from thoſe to whom he is unknown, he cannot expect more favour than his feeble efforts may deſerve.

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ELEMENTS

OF

ALGEBRA.

QUESTIONS in Common ARITHMETIC.

I SHALL here insert a Collection of promiscuous Questions relating to Numbers; and in solving them shall occasionally use Fractions, Logarithms, &c. Shall make very little Alterations of the Words in those Questions that have been published by others; but shall solve some of them by Methods different from any that I have yet seen.

1. Suppose a Person occupies 84l. per Annum in a certain Parish, and pays 14s. towards repairing the Church: What must another Person pay towards it, who occupies 197l. per Annum in the same Parish?

State it thus, As 84l. : 14s. :: 197l. : 1l. 12s. 10d. the Answer.

This stating is read thus, as 84l. is to 14s. so is 197l. to 1l. 12s. 10d. the fourth Number. For the third Number, viz. 197l. being multiplied by 14s. the second Number, the Product is 2758 Shillings, which divided by 84l. the first Number, gives 32s. and 70s. remain, which

being multiplied by 12, and the Product, viz. 840, divided by 84, the Quotient is 10d. so that the Answer is 32s. 10d. or 1l. 12s. 10d. as above.

2. Suppose a Person has an Estate of 200l. per Annum, and pays 2l. 1s. 8d. to a Subsidy, What is that in the Pound?

State it thus, as 200l. : 2l. 1s. 8d. : : 1l. : $2\frac{1}{2}$d. the Answer.

This stating is read thus, as 200l. is to 2l. 1s. 8d. so is 1l. to $2\frac{1}{2}$d. for 2l. 1s. 8d. the second Number being reduced it is 500 Pence, which divided by 200l. the first Number, gives $2\frac{1}{2}$d. for the fourth Number, or Answer as above.

These two Solutions being thus explained, the rest will appear as obvious to those who have a tolerable Notion of the Rule of Three. as if the Operations were wrought at large; for he must have but a very shallow Knowledge of Arithmetic, who cannot work a Question in the Rule of Three when it is already stated for him; and since it will not require half so much Room to insert the Solutions by this Method, as it would to set down all the Work at full length, I shall therefore solve the succeeding Problems (where it is convenient) in the same Manner as those above.

3. If the Valuation of a Parish be 1600l. per Annum, and if the Poor's Rate amount to 63l. 6s. 8d. per Quarter: What must a Person pay towards this Rate who occupies (by an equal Valuation) 125l. a Year in the said Parish, and what is the Rate per Pound?

Answer, the Person who occupies 125l. must pay 4l. 18s. $11\frac{1}{2}$d. and the Rate is $9\frac{1}{2}$d. in the Pound.

For as 1600l. : 63l. 6s. 8d. : : 125 : 4l. 18s. $11\frac{1}{2}$d. what this Occupier pays; and, as 1600l. : 63l. 6s. 8d. : : 1l. : $9\frac{1}{2}$d. the Rate per Pound.

Thus you may find what each Occupier must pay to any Assessment, Tax, &c. and what any Parish Charges amount to in the Pound.

4. A Bankrupt is indebted 2980l. 10s. but all his Effects amount but to 1117l. 13s. 9d. What have his Creditors

tors in the Pound, and what must that Man receive for his Part of the Dividend, whose Demand on the said Bankrupt is 400l.

First, as 2980l. 10s. : 1117l. 13s. 9d. : : 1l. : 7s. 6d. what the Creditors must have in the Pound:

And as 1l. : 7s. 6d. : : 400l. : 150l. his part of the Dividend whose Demand is 400l.

And in this Manner the particular Shares of every Creditor may be found, if their respective Demands be known.

5. Bought 6 Packs of Cloth, which contained 2240 Yards, and gave after the Rate of 42s. for three Yards; What did it cost me, and what shall I gain per Cent. by selling it out at 17s. 6d. per Yard.

First, 42 ÷ 3 = 14s. equals what 1 Yard cost; and 17s. 6d. — 14s. = 3s. 6d. my Gain by 14s. (or by selling 1 Yard.)

Then, state it thus, as 1 Yard : 14s. : : 2240 Yards : 1568l. the Purchase of the Cloth.

And as 14s. : 3s. 6d. : : 100l. : 27l. the gain per Cent.

If you would know at what Rate you must sell out your Goods by retail, so as to make a proposed Gain by the Whole, add the Money you would gain to the Sum which the whole Goods cost you: But if at any Time some Damage having happened to the Goods, so as to make a proposed Loss by the Whole, then subtract the said Loss from the Cost, and make the Remainder the second Number. An Example in each Case will make this quite easy.

6. A Grocer bought 11 Hogsheads of Sugar, which weighed net 34 cwt. 1 qr. 4 lb. for 64l. and would gain 32l. by the Bargain: At what Rate must he sell this Sugar per lb.

First, 64l. + 32l. = 96l. what he sells the whole for; and, as 32 cwt. 1 qr. 4 lb. or 3840lb. : 96l. : : 1 lb. : 6d. the Answer.

7. A Draper

7. A Draper bought 400 Yards of broad Cloth for 230l. but it having received Damage, he is willing to ſell it ſo as to loſe 30l. by the Whole; how muſt he ſell it per Yard?

Firſt, 230l. — 30l. = 200l. what he ſells it for:
And as 400 yds. : 200l. : : 1 yd. : 10s. the Anſwer.

8. If 96 Ells of Cambric coſt 60l. how muſt it be ſold per Yard, to gain 25 per Cent.

Firſt, 96 Ells $\times \frac{5}{4} = \frac{480}{4} =$ 120 Yards.

And as 100l. : 60l. : : 125l. : 75l. what the 120 Yards muſt be ſold for to gain 25 per Cent.

And, as 120 yds. : 75l. : : 1 yd. : 12s. 6d. the Anſwer.

9. A Merchant bought 10 Tuns of Wine for 669l. 7s. 6d. but ſome Damage having happened to it, he intends ſelling it ſo as to loſe after the Rate of 20 per Cent. How muſt he ſell it per Gallon?

Firſt, as 100 : 669l. 7s. 6d. : : 80l. : 535l. 10s. what he muſt ſell it for, to loſe 20 per Cent:

And, as 10 Tuns, or 2520 Galls. : 535l. 10s. : : 1 Gall. : 4s. 3d. the Anſwer.

By proceeding as in theſe laſt four Solutions, you may determine how to ſell any Goods by retail, ſo as to make any propoſed Gain or Loſs by the Whole, or at any Rate per Cent.

10. Bought Threeſcore Pieces of Holland for three Times as many Pounds, and ſold them again for four Times as much; but if they had coſt me as much as I ſold them for, what ſhould I have ſold them for, to gain after the ſame Rate?

Firſt, 60 × 3 = 180l. what the 60 Pieces coſt:
And 60 × 4 = 240l. what they were ſold for:
And, as 180l. : 240l. : : 240l. : 320l. the Anſwer.

11. If by ſelling Hops at 3l. 15s. per Cwt. I clear 30 per Cent. what ſhall I gain per Cent. if I ſell the ſame Goods for 4l. 10s. the Cwt.

Firſt,

First, as 3l. 15s. : 130l. : : 4l. 10s. : 156l. the amount per Cent. by selling the Hops at 4l. 10s. per Cwt.

And therefore 156l. − 100l. = 56l. the Gain per Cent. required.

12. A Person delivered to another a Sum of Money unknown, to receive simple Interest for the same at 5 per Cent. per Annum, and at the End of 8 Years received for Principal and Interest 630l. What was the Sum lent?

First, $100 + 5 \times 8 = 140$l. the Amount of 100l. for 8 Years.

And, as 140l. : 100l. : : 630l. : 450l. the Answer.

13. In some Parishes in the Country, they take off 3l. a Year in 17l. from the Rents in assessing the Farms; what will the Landlord receive net out of a Farm of 204l. a Year, in those Places where the King's Tax is, as now, 4s. in the Pound?

First, as 17l. : 3l. : : 204l. : 36l. taken off; therefore 204l. — 36l. = 168l. is the Assessment, which, divided by 5 (because 4s. is $\frac{1}{5}$ of a Pound) gives 33l. 12s. for the Taxes; which, being taken from 204l. the Rent, leaves 170l. 8s. the Answer.

14. It is a Rule in some Parishes to assess the Inhabitants in Proportion to $\frac{4}{5}$ of their Rents: What is the Yearly Rent of that House, which pays 8l. 10s. to the King under this Limitation, at 4s. in the Pound?

First, as 4s. : 1l. : : 8l. 10s. : 42l. 10s. the Assessment.

And, as 4 : 42l. 10s. : : 5 : 53l. 2s. 6d. the Rent required.

N. B. This is the 36th Question in the Rule of Three, in a most excellent System of Arithmetic, second Edition, where, by Mistake, the Assessment is only found, and inserted for the Answer, instead of the Rent.

15. A Vintner bought a Pipe of Wine at 7s. 6d. per Gallon, with which he mixed a certain Quantity of Water, and sold the Mixture at 10s. per Gallon, his Gain

upon the Whole was 19l. 12s. 6d. How many Gallons of Water did he put in?

First, 10s. − 7s. 6d. = 2s. 6d. his Gain per Gallon by the Wine alone; and (126) the Gallons in a Pipe divided by 8, (because 2s. 6d. is $\frac{1}{8}$ of a Pound) gives 15l. 15s. for his Gain by the Wine, exclusive of the Water, this taken from 19l. 12s. 6d. the whole Gain, leaves 3l. 17s. 6d. for his Gain by the Water:

And, as 10s. : 1 Gall. : : 3l. 17s. 6d. : 7 Galls. 3 Qts. the Quantity of Water sought.

16. A Stationer sold Quills at 11s. per Thousand, by which he cleared $\frac{3}{8}$ of the Money; but growing scarce, raised them to 13s. 6d. per Thousand: What might he clear per Cent. by the latter Price?

First, $\frac{3}{8}$ of 11s. $= \frac{33}{8} =$ 4s. $1\frac{1}{2}$d. the Gain at 11s. per Thousand.

Therefore 11s. − 4s. $1\frac{1}{2}$d. = 6s. $10\frac{1}{2}$d is the prime Cost of a Thousand:

And therefore 13s. 6d. − 6s. $10\frac{1}{2}$d = 6s. $7\frac{1}{2}$d. is his Gain per Thousand by the latter Price.

And as 6s. $10\frac{1}{2}$d. : 6s. $7\frac{1}{2}$d. : : 100l. : 96l. 7s. $3\frac{1}{4}$d $\frac{1}{11}$ the Rate per Cent. required.

17. A Person bought a certain Number of Ells of Velvet, which he sold again; he bought 5 Ells for 7 Crowns, and sold 7 Ells for 11 Crowns, and gained 100 Crowns in so doing: I demand how many Ells there were in all?

First, as 5 Ells : 7 Crowns : : 7 Ells : $9\frac{4}{5}$ Crowns, what 7 Ells cost him; but he sold 7 Ells for 11 Crowns, therefore 11 − $9\frac{4}{5}$ = $1\frac{1}{5}$ Cr. = 6s. his Gain upon 7 Ells:

And, as 6s. : 7 Ells : : 100 Crowns (or 500s.) : $583\frac{1}{3}$ Ells the Answer.

18. Bought Stockings in London at 4s. 3d. the Pair, and sold them afterwards in Dublin at 6s. the Pair; now taking the Charges at an average to be 2d. the Pair, and considering that I must lose 12 per Cent. by remitting my

Money

Money Home again: What do I gain per Cent. by this Article of Trade?

First, 4s. 3d. + 2d. = 4s. 5d. the prime Cost and Charges; and 100l. — 12l. = 88l. what I received instead of 100l. on Account of the Remittance.

And, as 4s. 5d. : 6s. : : 88l. : 119l. 10s. $11\frac{1}{4}$d. $\frac{15}{53}$ the return per 100l. therefore 119l. 10s. $11\frac{1}{4}$d $\frac{15}{53}$ — 100l. = 19l. 10s. $11\frac{1}{4}$d. $\frac{15}{53}$ the Rate per Cent. required.

Otherwise thus:

As, 4s. 5d. : 6s. : : 100l. : 135l. 16s. $11\frac{3}{4}$d. $\frac{5}{53}$ (the Amount of 100l.) whose Interest at 12 per Cent. is 16l. 6s. $0\frac{1}{4}$d. $\frac{43}{53}$ this added to 100l. and the Sum taken from 135l. 16s. $11\frac{3}{4}$d. $\frac{5}{53}$, leaves 19l. 10s. $11\frac{1}{4}$d. $\frac{15}{53}$ for the Gain per Cent. the same as before.

19. A Hare is 100 Yards distant from a Dog, and both starting together, the Dog ran $2\frac{1}{2}$ Times faster than the Hare: It is demanded how far the Hare will have run before the Dog overtakes her?

First, $2\frac{1}{2} - 1 = 1\frac{1}{2}$ Yard which the Dog gains of the Hare while she runs 1 Yard:

And, as $1\frac{1}{2}$ Yard : 1 : : 100 Yards, : $66\frac{2}{3}$ Yards, the Answer.

20. A Person being asked what Hour of the Day it was, answered, it is between 5 and 6, and both the Minute Hand and Hour Hand are together: Required the Hour of the Day?

It is well known that the Minute Hand points to 12 at the End of every Hour, consequently it points to 12 at 5 o'Clock; but the Hour Hand goes only $\frac{1}{12}$ of the Circumference in an Hour, the other goes the whole round (viz. $\frac{12}{12}$) in that Time; therefore the Minute Hand gains of the other $\frac{12}{12} - \frac{1}{12}$, or $\frac{11}{12}$ of the Circumference in an Hour: But at 5 o'Clock the Hour Hand is $\frac{5}{12}$ of the Circumference before the Minute Hand: Therefore say, if the Minute Hand gains $\frac{11}{12}$ of the Circumference in an Hour, in what Time will it gain $\frac{5}{12}$ thereof?

State it thus, if $\frac{11}{12}$ Cir. : 1 Hour : : $\frac{5}{12}$ Cir. But, because the Denominations of the first and third Numbers

 are

are equal, they may therefore be rejected, and so it will be barely

As 11 : 1 Hour : : 5 : 27 $\frac{3}{11}$ Minutes past 5, the Time sought. For the next Conjunction it will be as 11 : 1 Hour : : 6 : 32 $\frac{8}{11}$ Minutes past 6; for the next it will be as 11 : 1 Hour : : 7 : 38 $\frac{2}{11}$ Minutes past 7.

Hence it follows that the Proportion for finding the next Conjunction after 12 o'Clock will be as 11 : 1 Hour : : 12 : 1 $\frac{1}{11}$ Hour, or 5 $\frac{5}{11}$ Minutes after one of the Clock, for those two Hands are always together at the Hour 12.

21. If the Sun moves every Day 1 Degree, and the Moon 13 and at a certain Time the Sun be at the beginning of Cancer, and in 3 Days after the Moon in the beginning of Aries: The Place of their next following Conjunction is required:

Note, From the beginning of Aries to the first of Cancer there are three Sines of the Zodiac included, viz. Aries, Taurus, and Gemini, and each Sine contains 30 Degrees; so that from the first of Aries to the first of Cancer are 90 Degrees, to which add 13 × 3, or 39 Degrees, the Space through which the Moon passes in 3 Days (in which Time she arrives at Aries) and you will have 129 Degrees for the Distance of the Sun before the Moon; But 13 — 1 = 12 Degrees the Moon gains of the Sun per Day, therefore, it will be,

As 12° : 1 Day : : 129° : 10 $\frac{3}{4}$ Days, in which Time the Sun will be overtaken by the Moon, and because the Sun moves 1 Degree every Day, therefore the Conjunction required is in 10 $\frac{3}{4}$ Degrees of Cancer.

Note, The Sun doth not describe 1 Degree in a Day completely, however, the foregoing Process is conformable to the Suppositions in the Question: And the Sun and Moon are said to be in conjunction with each other, when the Moon is exactly between the Earth and Sun; in which Case we cannot see her, but a little after her Conjunction with the Sun a small Portion of her Surface appears enlightened, and is therefore said to be a new Moon. Some Authors call the Conjunction itself a new Moon.

N. B. The Moon moves every Day about 13° 10′ from West to East.

22. The Globe of the Earth, under the Equinoctial Line, is 360 Degrees in Circumference; and this Body being turned on its own Axis in a natural Day, or 24 Hours, at what Rate an Hour are the Inhabitants of Bencoolen, (situated in the Midst of the torrid Zone) carried from West to East by this Rotation.

NOTE. The Equator lies directly in the midst of the torrid Zone, therefore (by the Question) Bencoolen is situated on the Equator, and any Circle whatever contains 360 Degrees; but the Length of a Degree of Longitude on the Equator, is 60 Geographical Miles, or 69 $\frac{1}{2}$ English Miles.

Therefore say, if a Point on this Circle moves 360 Degrees, or 25020 English Miles in 24 Hours; how far will it move in 1 Hour?

State it thus, as 24 Hours : 25020M. : : 1H : : 1042$\frac{1}{2}$M. the Answer.

But Bencoolen, according to some Authors, is situated in 4° South Latitude, however, the above Solution gives the Velocity per Hour of those who live on the Equator agreeable to the Problem. And it is in Degrees of the Equator that the Longitude of Places are reckoned; and as the natural Day is measured by one Revolution of the Equator, it follows that 1 Hour answers to 15 Degrees: For as 24 Hours : 360 Degrees : : 1 Hour : 15 Degrees: Hence 1 Degree of the Equator will contain 4 Minutes of Time; for 360 Degrees : 24 Hours : : 1 Degree, : 4 Minutes; 15 Minutes of a Degree will make 1 Minute of an Hour; for 360 : 24 Hours : : 15 Minutes : 1 Minute; and, 4 Seconds answer to 1 Minute of a Degree; for as 24 Hours : 360 Degrees : : 4 Seconds : 1 Minute of a Degree.

23. When the Sun is in the Meridian at Soho-Square, in what Time will it be so at Tyburn, lying due West of it at the Distance of a Geographical Mile, in the Latitude of 51 $\frac{1}{2}$ Degrees North, where a Degree of Longitude measures 37,35 (Geographical) Miles, known by the diurnal Rotation of the Earth to pass in 4 Minutes Times.

NOTE. Though a Degree of Longitude passes in 4 Minutes, yet the Measure of a Degree is determined from the Latitude of the Place and not from the

Time

Time in which it is described; for a Degree of Longitude is greater or less than 37,35 Miles in any other Latitude, except in 51 $\frac{1}{2}$ Degrees, North or South; hence, and from the Nature of the Question, it follows, that Soho-square and Tyburn are situated a Mile asunder, both on one Parallel of Latitude, which Parallel imagine to be a Circle on the Globe, and say, if a Point on this Circle moves 37,35 Miles in 4 Minutes, in what Time will another Point on the same Circle be moving 1 Mile with an equal Celerity?

State it thus, as, 37,35 Miles : 4 Minutes : : 1 Mile : 6 Seconds 25,542 Thirds +. So that it is Noon at Tyburn, 6 Seconds 25,542 Thirds after Twelve o'Clock at Soho-square.

24. There are 2 Bodies in Motion, viz. A and B; A has described 50 Miles; B only 5, but A has moved with 5 Times the Velocity of B; What is the Ratio then of the Times they have been describing those Spaces?

Here, since A's Motion was 5 Times as swift as B's, it is evident that in the Time B moved 5 Miles, A moved 25; but A moved 50 Miles in the Whole, therefore A was twice as long in Motion as B; for as 50 is to 25, so is 2 to 1.

25. A and B are two moving Bodies, and A moves 40 Times swifter than B, but A has been in motion but 1 Minute, whereas B has been in motion 2 Hours (or 120 Minutes:) The Ratio of the Spaces described by these 2 Bodies is required?

Here by the Question A was in Motion only 1 Minute, and A moved as far in one Minute as B did in 40; but B was three Times 40, or 120 Minutes in Motion; therefore the Space described by B is to that passed over by A, as 120 to 40, or as 3 to 1.

26. A Gentleman dying, left his Wife big with Child, ordering by Will, if the Child proved a Daughter, then his Wife should have $\frac{2}{3}$ and the Daughter $\frac{1}{3}$; but if it was a Son he should have $\frac{2}{3}$ and the Mother $\frac{1}{3}$ of the Estate. Now it happened that the Mother was delivered of a Son

Son and two Daughters; how muſt the Eſtate (which was 9000l.) be divided among them? Since a Daughter was to have $\frac{1}{3}$, and the Mother $\frac{2}{3}$, it is evident that the Mother muſt have as much as the 2 Daughters; and becauſe a Son was to have $\frac{2}{3}$ and the Mother $\frac{1}{3}$, it follows that the Son muſt have twice as much as his Mother, conſequently he muſt have as much as his Mother and Siſters: Therefore he muſt have $\frac{1}{2}$ of the Eſtate, viz. 4500l. The Mother muſt have $\frac{1}{4}$ of the Eſtate, viz. 2250l. and the Daughters muſt each have $\frac{1}{8}$ of it, which is 1125l.

27. A Farmer, ignorant in Numbers, ordered 500l. to be divided among his 3 Sons, thus: Give A, ſays he, $\frac{1}{3}$, B $\frac{1}{4}$, and C $\frac{1}{5}$ Part; divide this equitably among them, according to the Father's Intention.

Firſt, $\frac{1}{3}$, $\frac{1}{4}$ and $\frac{1}{5}$ of 500l, added together, make 391l. 13s. 4d. and, as 391l. 13s. 4d. : ($\frac{500}{3}$l, or) 166l. 13s. 4d. :: 500l. : 212l. 15s. 3$\frac{3}{4}$d. $\frac{15}{47}$ A's Part; then ſay, as $\frac{1}{3}$: 212l. 15s. 3$\frac{3}{4}$d. $\frac{15}{47}$ :: $\frac{1}{4}$: 159l. 11s. 5$\frac{3}{4}$d. $\frac{23}{47}$, B's part; and as $\frac{1}{3}$: 212l. 15s. 3$\frac{3}{4}$d. $\frac{15}{47}$ :: $\frac{1}{5}$: 127l. 13s. 2$\frac{1}{4}$d. $\frac{9}{47}$ C's Part.

N. B. In the ſecond ſtating I multiplied the ſecond Number by 3, the Denominator of the firſt Number, and divided the Product by 4 and by 5, the reſpective Denominators of the third Numbers in the two laſt Statings, whence B's and C's Portions were produced by a very few Figures. But C's Part might have been found by ſubtracting the Sum of the Portions of A and B from 500l. the given Dividend. The Reaſon of the preceeding Operation is obvious; for ſeeing that the firſt two Numbers in the ſecond and third Statings are the ſame, it is evident that the Products of thoſe two Numbers, in each of theſe Statings, will be equal to each other, and in all ſimilar Caſes, having found one of the Parts, the reſt may be readily obtained by proceeding as above.

28. A and B for the ſame Time made a joint Stock of 700l. by which they gained 160l. of which A received 32l. more than B; what was each Perſon's Stock?

Firſt,

First, A received $\frac{160+32}{2}=96$l. then, as, 160l. : 700l. : : 96l. : 420l. A's Stock, and 700 — 420 = 280l. B's Stock.

29. A Lady was asked her Age, who replied thus:

My Age if multiplied by three,
Two sevenths of that Product tripled be,
The Square Root of two Ninths of that is four;
Now tell my Age, or never see me more.

Suppose the Lady was 21 Years of Age: Then multiplying 21 by 3, the Product is $21 \times 3 = 63$, and two sevenths of the Product is $63 \times \frac{2}{7} = 18$; which, being tripled or multiplied by 3, the Product is $18 \times 3 = 54$, and two ninths of this Product is $54 \times \frac{2}{9} = 12$, which ought to have been 16, and then its square Root would have been 4, agreeable to the Conditions of the Question.

Hence it will be, as 12 : 21 : : 16 : 28 Years, the Age required.

30. A Man skaited against the Wind 8 Miles in an Hour and 20 Minutes, returning with an uniform Stroke, having then the Advantage of an equal Wind, he skaited back the same Distance in 32 Minutes: Required the Force of the Wind.

First, find how far he skaited both with and against the Wind in any Particle of Time (as 1 Minute) and half the Difference of those Velocities will be the effective Increase and Decrease of his Celerity in that Time, occasioned by the Wind.

Thus, $5280 \times 8 = 42240$ Feet in 8 Miles;

Then, as 80 Minutes : 42240 Feet, : : 1 Minute : 528 Feet, his Velocity per Minute against the Wind;

And, as 32 Min. : 42240 Feet, : : 1 Min. : 1320 Feet, his Celerity per Minute with the Wind:

Therefore his Celerity was augmented and retarded by the Wind $\frac{1320-528}{2}$ or 396 Feet per Minute.

31. If a Man travels 25 Miles a Day, and every Night comes back in the same Road 6 Miles; How many Days will

will he be travelling 386 Miles from the Place at which he first sets out?

From the given Distance (viz. 386 Miles) subtract the Space through which he returns in a Night, and divide the Residue by the Space gained in a Day, the Quotient is the Answer, if nothing remains; if there be a Remainder add to it the Space he comes back in a Night, and divide that Sum by the whole Distance travelled in a Day, the Quotient will give the Time in which he must travel the last Day. But in the Case proposed there is no Remainder of the Division, for $25 - 6 = 19$ Miles, the Space gained in a Day; and $386 - 6 = 380$ Miles, the Residue of the given Distance: Then $380 \div 19 = 20$ Days, the Time required.

The Ground of this Solution will appear very obvious, if it be considered that he gains 25 Miles in the last Day; for in the last Day he gets to his Journey's End.

32. There is a Sum of 1000l. to be divided among three Men, in such Manner, that if A has 3l. B shall have 5l. and C 8l. how much must each Man have.

First, $3 + 5 + 8 = 16$, the Sum of the Numbers denoting the respective Proportions of their Shares; and 1000l. $\div 16 = 62{,}5$, therefore A must have $62{,}5 \times 3 =$ 187,5l. B $62{,}5 \times 5 =$ 312,5l. and C $62{,}5 \times 8 =$ 500l.

33. If A can do a Piece of Work in 20 Days, and B in 30 Days; in what Time will it be finished by A and B working together?

Let the Work be denoted by Unity or 1; then since A can do $\frac{1}{20}$ and B $\frac{1}{30}$ Part of it in a Day, it is evident that they both together can do $\frac{1}{20} + \frac{1}{30} = \frac{5}{60} = \frac{1}{12}$ Part of it in a Day; and, as $\frac{1}{12}$: 1 Day :: $\frac{1}{1}$: 12 Days, the Answer.

34. A Person in the last War was possessed of $\frac{4}{5}$ of a Privateer; he sold $\frac{3}{4}$ of his Interest therein for 400l. what was the Value of the whole Privateer?

First, $\frac{3}{4}$ of $\frac{4}{5} = \frac{12}{20} = \frac{3}{5} =$ three fourths of his Interest, which, by the Question, is worth 400l. therefore it will be, as $\frac{3}{5}$: 400l. :: $\frac{1}{1}$: 666l. 13s. 4d. the Answer.

35. Two

35. Two Persons purchased an Estate of 1700l. per Annum Freehold for 27200l. when Money was at 6 per Cent. Interest, and 4s. per Pound Land-Tax, whereof D paid 15000l. and E the rest; some Time after the Interest of Money falling to 5 per Cent. and 2s. per Pound Land-Tax, they sell the said Estate for 24 Years purchase; I desire to know each Person's Share?

First, $1700 \times 24 = 40800$l. what they sell the Estate for; and as 27200l. : 40800l. : : 15000l. : 22500l. D's Share; therefore 40800l. — 22500l. = 18300l. E's Share.

36. Sold a Quantity of Muslin for 588l. and by so doing lost 16 per Cent. whereas in Dealing I ought to have cleared 25 per Cent; how much then was it sold under the just Value?

Here it is to be observed, that for every 100l. which the Muslin cost, I received in return (by selling it 100 — 16 or) 84l. therefore, as 84l. : 125l. : : 588l. : 875l. what the Muslin should have been sold for to gain 25 per Cent. therefore 875l. — 588l. = 287l. the Answer.

37. Suppose the Wheel of a Carriage by its Rotation to describe a right Line on the Ground equal to its Circumference; and, admitting that the circular Velocity of a certain Wheel is to that of its Nave, as 2 to $\frac{1}{3}$, and that the Circumference of the Wheel exceeds that of its Nave 5 Yards: Required how many Revolutions this Wheel will make in going 150 Miles?

Suppose the Circumference of the Wheel to be 12 Yards, then that of the Nave will be 2 Yards; for 12 is to 2 as 2 is to $\frac{1}{3}$; but the Difference of 12 and 2 is 10, instead of 5; hence, as 10 : 12 : : 5 : 6 Yards, the Circumference of the Wheel: And $1760 \times 150 = 264000$, the Yards in 150 Miles; which, being divided by 6, gives 44000, for the Number of Turns or Rotations required.

38. There is a Cannon of cast Iron, whose Weight is $65\frac{3}{4}$ Hundreds, it is required to find how many cubic or solid Feet of yellow Fir, lashed to the above Cannon, will be sufficient to keep it afloat in the Sea?

First,

First, $65,75 \times 112 = 7364$ lbs. the Weight of the Cannon, and as ,00258064 lb. the Weight of an Inch of cast Iron, is to 1 solid Inch thereof, so is 7364lbs. to 28535 cubic Inches, the Solidity of the Cannon: Now a Body heavier than Water weighs less in Water than in Air, by the Weight of so much Water as is equal in Bulk to the Body, therefore, as 1 cubic Inch of Sea Water, is to ,037253 lbs. its Weight, so is 28535 cubic Inches (the solid Content of the Cannon) to 1063 the Weight of a Quantity of Sea Water, which is equal in Bulk to the Cannon: Therefore $7364 - 1063 = 6301$ lb. the remaining Weight of the Cannon, which is to be kept afloat, or bouyed up by the Fir. But a Body lighter than Water will sustain therein, the Difference between the Weight of the Body and that of an equal Bulk of Water; therefore, from ,037253 lbs. the Weight of 1 cubic Inch of Sea Water, take 0,023763, the Weight of a cubic Inch of Fir, and you will have 0,01349 lb. for the Weight which one cubic Inch of Fir will sustain in the Sea; consequently one cubic Foot will sustain $0,01349 \times 1728 = 23,31072$ lbs. and, as 23,31072 lbs. : 1 Foot : : 6301 lbs. : 270,3048 +, cubic Feet, the Quantity of Fir sought.

39. If a Bath Stone 20 Inches long, 15 broad, and 8 Inches thick, weighs 220lbs. how many cubic Feet thereof will freight a Ship of 290 Tons Burthen?

First, $20 \times 15 \times 8 = 2400$, solid Inches in 1 Stone, which, by the Question, weighs 220 Pounds, therefore as 2400 Inches : 220lbs. : : 1728 Inches : 158,4 lb. the Weight of one cubic Foot of the Stone; and as 158,4lbs. : 1 Foot : : 290 Tons : 4104, Feet the Answer.

40. A Hare pursued by a Dog, takes five Leaps for the Dog's four; but the Dog takes as much at three Leaps as the Hare doth at four. Now the Hare being eight Hundred of her own Leaps before the Dog at starting; how many Leaps must each take during the Course?

Since the Dog takes as much at 3 Leaps as the Hare does in 4, it is evident that $\frac{3}{4}$ of a Leap of the Dog's, make 1 Leap of the Hare's, and consequently 800 of the Hare's Leaps are equal to 600 of the Dog's; so that the Hare

Hare was 600 of the Dog's Leaps before him at starting; and, as 1 : $\frac{3}{4}$ :: 5 : $3\frac{3}{4}$ Leaps of the Dog's, which are equal in Distance to 5 of the Hare's; but the Dog ran 4 Leaps to the Hare's 5, therefore the Dog gained $4-3\frac{3}{4}$ or $\frac{1}{4}$ of a Leap on the Hare in running 4 Leaps; hence, as $\frac{1}{4}$: 4 :: 600 : 9600 Leaps made by the Dog; and, as 4 : 9600 :: 5 : 12000 Leaps ran by the Hare.

41. There is an Island 73 Miles round, and 3 Footmen all start together, to travel the same Way about it; A travels 5 Miles a Day; B 8, and C 10; when will they all come together again?

First, $10-8=2$ Miles C gained of B in one Day, and, as 2 Miles : 1 Day :: 73 Miles : 36,5 Days, in which B and C will meet for the first Time after their setting out; therefore they have not all met yet, unless they be all together now, which is impossible, because A's Velocity is not equal to B's, nor to C's; but A goes half as swift as C, therefore A will be at the Place where B and C first met, in 36,5 Days more, in which Time and Place it is evident that B and C will meet again; and consequently they will be all together in (36,5 + 36,5, or) 73 Days.

42. Sold a certain Quantity of Shalloon for 320l. by which I gained after the Rate of 33l. $\frac{1}{3}$ per Cent. but if I had sold it for 2s. 8d. per Yard, I should have lost 8l. 10s. per Cent. How many Yards did I sell?

First, as 133l. $\frac{1}{3}$: 100l. :: 320l. : 240l. the prime Cost, and 100l. − 8l. 10s. = 91l. 10s. the Return per 100l. if it had been sold for 2s. 8d. per Yard; then as 100l. : 240l. :: 91l. 10s. : 219l. 12s. what the Shalloon would fetch at 2s. 8d. per Yard; and as 2s. 8d. : 1 Yard, :: 219l. 12s. : 1647 Yards, the Answer.

43. A Gentleman hired a Servant for 12 Months, and agreed to allow him 20l. and a Livery, if he staid till the Year was expired; but at the End of 8 Months the Servant went away and received 12l. and the Livery, as a proportional Part of his Wages: What was the Livery valued at?

As

As 8 is $\frac{2}{3}$ of 12; therefore 20 × $\frac{2}{3}$ = 13l. $\frac{1}{3}$ his Wages for 8 Months; and 13 $\frac{1}{3}$ − 12 = 1l. $\frac{1}{3}$, discount for the Livery for 4 Months (or $\frac{1}{3}$ of the Year which he should have served, which is therefore $\frac{1}{3}$ of the Value of the Livery, and therefore 1 $\frac{1}{3}$ × 3 = 4l. the Answer.

44. If 5 Oxen or 7 Colts will eat up the Grass of a Close in 87 Days; in what Time will 2 Oxen and 3 Colts eat up the same.

If 5 Oxen be equivalent in feeding to 7 Colts, then 2 Oxen will be equal therein to 2 $\frac{4}{5}$ Colts; to this add 3 Colts; and you will have 5 $\frac{4}{5}$ Colts; which will eat as much as 2 Oxen and 3 Colts; and, as 7 Colts : 87 Days : : 5 $\frac{4}{5}$ Colts : 105 Days, the Answer.

Note. This stating is inverse, for it is evident that the fewer Colts there are the longer Time they will require to eat the same Quantity.

45. What is the Difference between the Interest and Discount of 500l. for 12 Years, at 5 per Cent. per Annum.

First, as 100l. : 5l. : : 500l. : 25l. the Interest of 500l. for 1 Year, and 25 × 12 = 300l. its Interest for 12 Years: But 5 × 12 = 60l. is the Interest of 100l. for 12 Years; and as 160l. : 60l. : : 500l. : 187l. 10s. the Discount for that Time; therefore 300l.—187l. 10s. = 112l. 10s. the Difference required.

46. A certain Sum of Money was lent at simple Interest, which in 8 Months amounted to 297,6, and in 15 Months to 306l. I demand the Principle and Rate of Interest.

First, 306l.—297,6 = 8,4l. the Interest of the Sum lent for (15—8, or) 7 Months; and as 7 Months : 8,4l. : : 15 Months : 18l. the Interest of the same Sum for 15 Months; therefore 306—18 = 288l. the Sum lent: Now say, if 288l. in 7 Months, gain 84l. Interest, what will 100l. gain in 12 Months?

State it thus: If 288l.— 7 Months——84l.
100 — 12 ——— * Here 288 × 7 = 2016, the Divisor, and 8,4 × 12 × 100 = 10080, the

the Dividend therefore 10080 ÷ 2016=5l. the Rate per Cent. required.

47. A Gentleman lent a Sum of Money at 5 per Cent. per Annum, ſimple Intereſt, for 10 Years, at the End of which Time the Intereſt amounted to 360l. I demand the Principal.

State it thus, If 100l. —— 1 Year —— 5l.
* —— 10 —— 360. Here 10 × 5 = 50 the Diviſor, and 360 × 100 = 36000 the Dividend, then 36000 ÷ 50=720l. the Principal ſought.

48. In what Time will 475l. amount to 646l. at 4 per Cent. per Annum.

Firſt, 646—475=171l. Intereſt of 475l. for the Time ſought.

State it thus, If 100l. —— 1 Year 4l.
475 —— * —— 171l. Here 475 × 4 = 1900, the Diviſor, and 171 × 100 = 17100, the Dividend, then 17100 ÷ 1900 = 9 Years the Anſwer.

49. A and B barter, A has 140lb. 11oz. of Plate, at 6s. 4d. the Ounce, which in truck he rates at 7s. 2d. an Ounce, and allows a Diſcount on his Part, to have $\frac{1}{7}$ of that in ready Specie; B has Tea worth 9s. 6d. the lb. which he rates at 11s. 2d. When they come to ſtrike the Balance, A received but 7 cwt. 2 qr. 18lb. of Tea, pray what Diſcount did A allow B, which of them had the Advantage, and how much, in an Article of Trade thus circumſtanced?

Firſt, 140 lb. 11 oz. = 1691 oz. and 7s. 2d.—6s. 4d. =10d. A's gain per Ounce; then, as 1 oz. : 10d. : : 1691 ozs. : 70l. 9s. 2d. the whole Advantage of A's. Plate; and as 1 oz. : 7s. 2d. : : 1691 oz. : 605l. 18s. 10d what A ſold the Plate for, $\frac{1}{7}$ of this is 605l. 18s. 10d. ÷ 7=86l. 11s. 3d. $\frac{1}{7}$, which A receives in ready Money; and the remaining Value for B is 605l. 18s. 10d.—86l. 11s. 3d. $\frac{1}{7}$, or 519l. 7s. 6d. $\frac{6}{7}$; but 7 cwt. 2 qr. 18lb. of Tea at 11s. 2d. per lb. comes to but 479l. 1s. therefore 519l. 7s. 6d. $\frac{6}{7}$—479l. 1s.=40l. 6s. 6d. $\frac{6}{7}$, is the Diſcount allowed by A: and B cleared 20d. per lb. by the Tea, therefore

therefore 7 cwt. 2 qr. 18lb. or 858 lb. divided by 12 (because 20d. is $\frac{1}{12}$ of a Pound) gives 71l. 10s. for what B gained by the Tea; to which add the Discount, and from that Sum take (70l. 9s. 2d) A's gain by the Plate, and there will remain 41l. 7s. 4d. $\frac{6}{7}$ for the Advantage to B.

50. If A raises Goods worth 30s. to 40s. and gives 9 Months Credit, how high must B raise Goods worth 27s. he agreeing to give 10 Months Credit.

First, 40s.—30s.=10s. A gains by 30s. in 9 Months; and in Trade, Time is proportional to Stock, therefore say, if 30s. in 9 Months gain 10s. what will 27s. gain in 10 Months: state it thus:

If 30s.——9 Months ———10s.

27——10 ——— *. Here $30 \times 9 = 270$ the Divisor, and $27 \times 10 \times 10 = 2700$ the Dividend, then $2700 \div 270 = 10$s. B's gain by 27s. in 10 Months, therefore B must raise his Goods to (27 + 10 or) 37s.

51. Having bought Goods for 600l. and sold them directly for 630l. giving 4 Months Credit, what is gained per Cent. per Annum.

First, 630l.—600l. = 30l. gained by 600l. in 4 Months, and as 600l. × 4 M. : 30l. : : 100l. × 12 M. : 15l. per Cent. the Answer.

52. Suppose A has 24 $\frac{1}{2}$ Yards of broad Cloth at 14s. per Yard, but in barter with B will have 16s. 4d. and $\frac{1}{4}$ of the whole Barter Value in ready Money; now if B's Cloth be worth 16s. 4d. per Yard ready Money, how many Yards of Cloth must B deliver to make the Barter equal.

First, from both the ready Money and Barter Value of A's Cloth, take $\frac{1}{4}$ of A's barter Price; and as the first Remainder is to the second, so is B's ready Money Price to his barter Price.

Thus $\left\{\begin{matrix}14\text{s. }0\text{d.}—\\16\text{s. }4\text{d.}—\end{matrix}\right\}$ 16s. 4d. ÷ 4 $\left\{\begin{matrix}=9\text{s. }11\text{d.}\\=12\text{s. }3\text{d.}\end{matrix}\right\}$ then as 9s. 11d. : 12s. 3d. : : 16s. 4d. : 242d. $\frac{2}{17}$ B's barter Price, and $24\frac{1}{2} \times \frac{3}{4} = 18\frac{3}{8}$ Yards, equal to $\frac{3}{4}$ of A's Cloth, and as 16s. 4d. : 18 $\frac{3}{8}$ Yards : : 242d. $\frac{2}{17}$: 14$\frac{7}{8}$ Yards, the Answer.

The ſame otherwiſe without finding B's barter Price. Here $\frac{1}{4}$ of $24\frac{1}{2}$ Yards, at 16s. 4d. per Yard, comes to 5l. 0s. 0d. $\frac{1}{2}$ and as 14s. : 1 Yard : : 5l. 0s. 0d. ^{1} : $7\frac{7}{48}$ Yards, the real Quantity of A's Cloth, which B muſt have for (5l. 0s. 0d. $\frac{1}{2}$) the ready Money; therefore $24\frac{1}{2} - 7\frac{7}{48} = 17\frac{17}{48}$ Yards, is the remaining Quantity, and as 14s.: 17 Yards $\frac{17}{48}$: : 16s. 4d. : $14\frac{7}{8}$ Yards, the Anſwer as before.

N. B. The laſt ſtating in each of theſe two Solutions is inverſe.

53. Three Merchants A B and C traded together; A puts into their joint Stock 400l. for 6 Months, B 600l. for 5 Months, and C 1000l. for 8 Months; with this Stock they gained 335l. it is required to find each Perſon's Share of the Gain, proportionable to his Stock and Time of employing it.

Firſt, $400 \times 6 + 600 \times 5 + 1000 \times 8 = 2400 + 3000 + 8000 = 13400$, the Sum of the Products of their ſeveral Stocks multiplied by their reſpective Times.

Then, as 13400l. : 335l. : : 2400 : 60l. the Share of A.

And, as 13400 : 335 : : 3000 : 75, the Share of B.

Alſo, as 13400 : 335 : : 8000 : 200 Share of C.

B's and C's Parts might have been obtained thus, as 4 : 60 : : 5 : 75 and as 3 : 75 : : 8 : 200, or C's Share might have been found by ſubtracting the Sum of A's and B's Shares from the whole Gain; and ſometimes the Shares may be more readily found by dividing the Gain by the Sum of the Products of their Stock aud Time, and multiplying the Quotient by each Man's particular Product. Thus, $335 \div 13400 = ,025$, then $025 \times 2400 = 60$l. A's Share, $,025 \times 3000 = 75$l. B's, and $,025 \times 8000 = 200$l. C's, the very ſame as before.

54. D owes E 800l. whereof 200l. is to be paid in 3 Months, 200l. at 4 Months, and 400l. at 6 Months; but they agreeing to make but one Payment of the Whole, at the Rate of 5 per Cent. rebate, the true equated Time is demanded?

First,

First, find the present Worth of each Payment for its proposed Time;

$$\text{Thus, as}\left\{\begin{array}{l} \text{£. s. d. £. £. £.} \\ 101\ \ 5\ 0 : 100 :: 200 : 197,53086+, \\ 101\ 13\ 4 : 100 :: 200 : 196,72131+, \\ 102\ 10\ 0 : 100 :: 400 : 390,\ \ 2439+, \end{array}\right\}\text{Present Worths}$$

The Sum of the present Worths is 784,49607, which being taken from 800l. leaves 15,50393 for the Discount, then by Question 48, it will be,

As 100l. ——— 12 Months ——— 5l.

784,49607 * ——— 15,50393. Here $784,49607 \times 5 = 3922,48035$, the Divisor, and $15,50393 \times 12 \times 100 = 18604,71600$, then $18604,71600 \div 3922,48035 = 4,743$ Months, that is, 4 Months, 22 Days, the Time required.

55. A Person having an Annuity left him for 4 Years, which does not commence till the End of 5 Years, disposed of it for 530l. allowing 4 per Cent to the Purchaser, what was the Yearly Income?

First, the Amount of 530l. for 5 Years, at 4 per Cent. is $530 + \frac{530}{100} \times 4 \times 5 = 636$l. and the Amount of 636l. for (4 Years) the Time of its Continuance, is 737,76l. But $4 + ,04 + ,08 + ,12 = 4,24$l. is the Principal and Interest of 1l. for 4 Years; and, as 4,24l. : 1l. :: 737,76l. : 174l. the Answer.

N. B. If the Payments be many, you may readily find the Interest of 1l. by multiplying the Sum of the Extremes by Half the Number of Terms.

56. How much must be put at Use for 2 Years at 5 per Cent. per Annum, to gain 80l.

State it thus,

If 100l. —— 1 Year —— 5l.

* 2 ——80l. then $\frac{100 \times 1 \times 80}{2 \times 5} = \frac{8000}{10}$

= 800l. the Answer.

57. What will 1230l. 17s. 6d. or 1230,875, amount to in 7 Years at 5 per Cent. per Annum compound Interest?

 First,

First, as 100l. : 105l. : : 1l. : 1,05l. the Amount of 1l for 1 Year, and $\overline{1,05}|^7$ = 1,4071004 +, its Amount for 7 Years, therefore 1230,875 × 1,4071004 = 1731,9647 +, = 1731l. 19s. 3d. $\frac{1}{2}$ + the Amount required.

58. A Broker bought for his Principal in the Year 1770, 400l. capital Stock in the South Sea at 650l. per Cent. and sold it when it was worth but 130l. per Cent. how much was lost in the Whole.

First, 650l. — 130l. = 520l. lost per Cent. and as 100l. : 520l. : : 400l. : 2080l. lost by the Whole.

59. If by remitting to Holland at 31s. 9d. Flemish per Pound Sterling 5 per Cent. is gained; how goes the Exchange when by Remittance I clear 10 per Cent.

First, as 105l. : 100l. : : 31s. 9d. : 30s. $\frac{5}{21}$ Flemish, the intrinsic Value of 1 Pound sterling; and as 100l. : 110l. : : 30s. $\frac{5}{21}$: 1l. 13s. 3d. $\frac{1}{7}$, the Answer.

60. If $\frac{1}{3}$ of 6 be 3, what will $\frac{1}{4}$ of 20 be?

First, $\frac{1}{3}$ of 6 is 2, $\frac{1}{4}$ of 20 is 5; and as 2 : 3 : : 5 : 7$\frac{1}{2}$, the Answer.

61. If 74 Yards of English be equal to 100 Brasses of Florence, and 100 Brasses of Florence be equal to 30 Canes of Marseilles; how many Canes of Marseilles are equal to 100 Yards of English.

Here, because 74 Yards are equal to 100 Brasses, and 100 Brasses equal to 30 Canes; therefore the Brasses may be rejected in the Operation, and it will be barely as 74 Yards English : 30 Canes : : 100 Yards Eng. : 40 $\frac{20}{37}$ Canes, the Answer.

62. If 40 lbs. at London make 36 lbs. at Amsterdam, and 90 lbs. at Amsterdam make 116 lbs. at Dantzick, how many lbs. at London are equal to 130 lbs. at Dantzick?

First, as 36 lbs. : 40 lbs. : : 90 lbs. : 100 lbs. at London, which are equal to 116 lbs. at Dantzick (because 90 lbs. at Amsterdam are equal to 116 at Dantzick) and 116 lbs. : 100l. : : 130 lbs. : : 112 lbs. $\frac{2}{29}$ the Answer.

63. A Baker

63. A Baker purchased 22 Sacks of Flour, each containing 280 lbs. 7 Sacks he bought at 25s, per Sack, 10 at 30s. and the rest at 37s. per Sack; this Flour he mingles all together, and desires to know how he must sell it per Stone (or 14 lbs.) to gain 50l. per Cent. by the Mixture?

First, $25 \times 7 + 30 \times 10 + 37 \times 5 = 660$ Shillings, the prime Cost of the whole Mixture; and 280 lbs. $\times$ 22 = 6160 lbs. = 440 Stones, in the 22 Sacks, then as 440 Stones : 660 Shillings :: 1 Stone : 1s. 6d. the Prime Cost per Stone; and as 100l. : 150l. :: 1s. 6d. 2s. 3d. per Stone, the Answer.

64. Hiero, King of Sicily, ordered his Jeweller to make him a Crown, containing 63 Ounces of Gold; the Workmen thought substituting Part Silver therein, to have a proper Perquisite, which taking Air, Archimedes was appointed to examine it, who, on putting it into a Vessel of Water found it raised the Fluid, or that itself contained 8,2245 Cubic Inches of Metal, and having discovered that the Cubic Inch of Gold more critically weighed 10,36 Ounces, and that of Silver but 5,85 Ounces, he, by Calculation, found what Part of his Majesty's Gold had been changed, and you are desired to repeat the Process?

First, $63 \div 5,85 = 10,76923$ solid Inches, in 63 Oz. of Silver, and $63 \div 10,36 = 6,08108$, in ditto of Gold.

Their Difference is 4,68815; but the Difference betwixt the Solidity of the Crown, and that of 63 Ounces of Silver is (10,76923—8,2245, or) 2,54473, Cubic Inches; and as 4,68815 : 63 Ounces :: 2,54473 : 34,1964 Ounces of Gold, and consequently 63—34,1964 =28,8036 Ounces of Silver.

65. The battering Ram of Vespasian weighed, suppose 100000 lbs. and was moved, let us admit, with such a Velocity by Strength of Hands, as to pass through 20 Feet in one Second of Time, and this was found sufficient to demolish the Walls of Jerusalem; with what Velocity must a Bullet that weighs but 30 lbs. be moved, in order to do the same Execution?

Note. The Momentum, or Force, with which a moving Body will ſtrike any Obſtacle that lies in its Way, is equal to the Velocity of its Motion multiplied into its Quantity of Matter, and allowing that the Weights of Bodies are reſpectively proportionable to their Quantities of Matter (as it is aſſerted they are) it follows, that the Product of the required Velocity of the Bullet multiplied into its Weight, muſt be equal to the Weight of the battering Ram, multiplied into its given Velocity; hence it will be reciprocally, as 100000 lbs. : 20 Feet :: 30 lbs. : 66666 $\frac{2}{3}$ Feet per Second, the Velocity of the Bullet as required.

66. Taking a Walk on a calm Day, and ſeeing the Flaſh of a Piece of Ordnance the Inſtant of its going off at Languard Fort, and hearing its Report 40 Seconds afterwards; required the Diſtance from the Place of Obſervation to the ſaid Fort, admitting the Velocity of Sound (as it has been found by Experiments) to be uniformly at the Rate of 1142 Feet in 1 Second of Time?

Firſt, as 1″ : 1142 Feet :: 40″ : 45680 Feet, which being divided by 5280, the Feet in a Mile, gives 8 Miles 5 Furlongs $\frac{7}{33}$, the Anſwer.

In the very ſame Manner you may meaſure the Diſtance of the Clouds producing Thunder and Lightning; for ſuppoſe from the Inſtant we obſerve the Flaſh to the Moment we hear the Stroke of Thunder, we number 4 Seconds, then it is plain the Sound has come 4 Times 1142, that is 4568 Feet, or ſomewhat above $\frac{3}{4}$ of a Mile; and ſo far is the Diſtance of the Cloud.

In like Manner, the Diſtance of Ships on the Sea is known by the firing of Guns.

67. If a Body on the Earth weighs 3600 Pounds, what will be its Weight if it be elevated 10000 Miles above its Surface, ſuppoſing the Earth's Semi-diameter to be 4000 Miles?

Here the aſſigned Diſtance of the Body is 10000 + 4000 = 14000, Miles from the Earth's Centre, equal to 14000 ÷ 4000 = 3,5 Semi-diameters of the Earth from its Centre; but the Body when on the Earth's Surface is one Semi-diameter

Semi-diameter from its Centre, and the attracting Power of Bodies decreaſes univerſally as the Square of the Diſtance from their Centres increaſes, this being granted, it will be reciprocally as 1 : 3600 lbs. : : $\overline{3,5}|^2$, that is, as 1 : 3600 lbs. : : $12\frac{1}{4}$: $293\frac{43}{49}$ Pounds, the Weight of the Body at the Height propoſed.

68. What is the leaſt common Multiple of the nine Digits, viz. 1, 2, 3, 4, 5, 6, 7, 8, 9.

Firſt, the leaſt common Multiple of 2 and 4, is $\frac{2}{2} \times 4$, or 4 (for 2 is the greateſt common Meaſure of 2 and 4) the leaſt Multiple of 4 and 6 is $\frac{4}{2} \times 6$, or 12; that of 12 and 3 is $\frac{3}{3} \times 12$ or 12, that of 12 and 8 is $\frac{8}{4} \times 12$, or 24, that of 24 and 9 is $\frac{9}{3} \times 24$, or 72; and becauſe the Reſt of the Digits, viz. 1. 5. 7. do not admit of a common Diviſor among themſelves, greater than Unity, nor yet with (72) the leaſt Multiple of the other ſix Figures, therefore $1 \times 5 \times 7 \times 72 = 2520$ is the Multiple required.

69. If 12 Oxen will eat $3\frac{1}{3}$ Acres of Graſs in 4 Weeks, and 21 Oxen will eat 10 Acres in 9 Weeks, how many Oxen will eat 24 Acres in 18 Weeks, the Graſs being allowed to grow uniformly?

Firſt, if 12 Oxen eat up $3\frac{1}{3}$ Acres in 4 Weeks, then 21 Oxen will eat $13\frac{1}{8}$ Acres in 9 Weeks: For as $12 \times 4 : 3\frac{1}{3}$ A. : : $21 \times 9 : 13\frac{1}{8}$ Acres; but 21 Oxen eat only 10 Acres in 9 Weeks, therefore, in 5 Weeks, 10 Acres become by the Growth of the Graſs equivalent to $13\frac{1}{8}$ Acres, and conſequently $13\frac{1}{8}-10$, or $3\frac{1}{8}$ Acres is the Increaſe upon 10 Acres in 5 Weeks; for it is to be obſerved that $3\frac{1}{3}$ Acres is the real Quantity which 12 Oxen eat in 4 Weeks without any Increaſe of the Graſs; ſo that the Time of Increaſe upon 24 Acres is only (18—4, or) 14 Weeks, therefore ſay, if the Increaſe of 10 Acres in 4 Weeks be $3\frac{1}{8}$ Acres, what will be the Increaſe on 24 Acres in 14 Weeks? Anſwer, 21 Acres; for, as $10 \times 5 : 3\frac{1}{8} : : 24 \times 14 : 21$ Acres, which, being added to 24, gives 45 Acres for the whole Quantity of Graſs, which is to feed a certain Number of Oxen 18 Weeks, and in order to find this Number, ſay, if 12 Oxen in 4 Weeks

Weeks eat $3\frac{1}{3}$ Acres; how many Oxen in 18 Weeks will eat 45 Acres: State it thus,

If 12 Oxen —— 4 Weeks —— $3\frac{1}{3}$ Acres
—— 18 —— 45 Hence $\frac{12\times4\times45}{18\times3\frac{1}{3}}=\frac{2160}{60}$
$=36$ Oxen, the Number sought.

70. Required the Length of a Pendulum which shall vibrate just 48 Times in a Minute?

Note. It has been found by Experiments, that a Pendulum 39,2 Inches long, in our Latitude vibrates 60 Times in 1 Minute; and that the Lengths of Pendulums are to one another reciprocally as the Square of the Numbers of Vibrations made in the same Space of Time; therefore say as the Square of 48, which is 2304, is to the Square of 60, which is 3600, so is 39,2 Inches, the Length of the Standard Pendulum to 61,25 Inches, the Length of the Pendulum required.

71. What Proportion is there between the Arpent of France, which contains 100 Square Poles of 18 Feet each, and the English Acre, containing 160 Square Poles of $16\frac{1}{2}$ Feet each, considering that the Length of the French Foot is to the English as 16 to 15.

First, $18 \times 18 \times 100 = 32400$ French Feet in an Arpent; and $16,5 \times 160 = 43560$ English Feet in an Acre. Now to find how many English Feet are equal to 32400, the Feet in an Arpent, it will be reciprocally as $\overline{16}|^2 : 32400 :: \overline{15}|^2$ that is, $256 : 32400 :: 225 : 36864$, the English Feet in a French Arpent: So that the English Acre is to the Arpent of France as 43560 to 36864, or as 605 to 512.

N. B. The Numbers 43560 and 36864 were respectively reduced to 605 and 512, by dividing each of them by 72, their greatest common Measure.

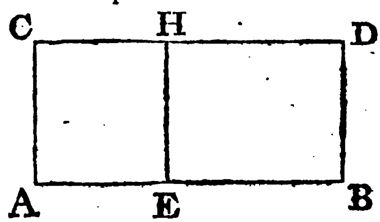
72. Suppose a Farmer covers a Field with Muck or Manure, and lays the Rows parallel to each other 8 Yards asunder, making 5 Heaps of a Load, and admitting the Distance from Heap to Heap in the Rows to be 6 Yards from their Centres; how many Loads will he lay on an Acre at this Rate,

First,

First, $8 \times 6 \times 5 = 240$ square Yards, the Area which 1 Load covers, and an Acre contains 4840 square Yards; therefore $4840 \div 240 = 20\frac{1}{6}$ Loads per Acre, the Answer.

73. It is required to divide the Parallelogram A B C D into two Parts, by a Line H E parallel to the Side A C, so as the Part A C E H shall contain just 11 Acres, when the Area of the whole Parallelogram is 27 A. 2 Roods, 21 Rods, and the Side A C is equal to 8 Chains.

First $11 \times 10 = 110$ square Chains in 11 Acres, and $110 \div 8 = 13{,}75$ Chains, or 55 Rods, equal to A E, (or C H) the Length sought.



74. If the internal Length and Breadth of a Rectangular Cooler be 94 and 40 Inches respectively, how deep must it be to contain 120 Ale Gallons?

First, $282 \times 120 = 33840$, the solid Inches in 120 Gallons, and $94 \times 40 = 3760$ Inches, the Solidity of the Cooler at 1 Inch deep; then as 3760 : 1 Inch :: 33840 : 9 Inches, the Depth required.

And by having any two Dimensions of a Cooler and its Solidity, the rest may be found by proceeding as above; the same may be observed of a rectangular Cistern, Tun, &c.

Thus, in the foregoing Example on the Cooler, it will be as 94×9, or 846 Inches, the Area of 1 Side is to 1 Inch, so is 33840 Inches, its Solidity, to 40 Inches, its Breadth; and as 40×9, or 360 Inches the Area of 1 End is to 1 Inch, so is 33840 Inches to 94 Inches, its Length.

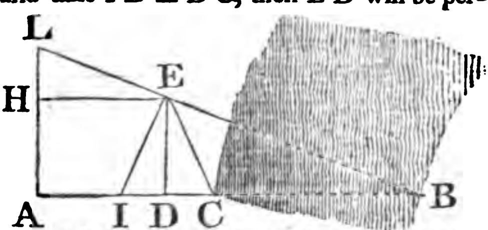
75. If the Diameter of a Cylinder's Base be 34 Inches, what must be its Length to contain 30 solid Feet?

Note, 0,7854 is the Area of a Circle whose Diameter is 1, and if the Square of the Diameter of any Circle be multiplied

multiplied by ,7854, the Product will be its Area. Thus, $34 \times 34 \times ,7854 = 907,9224$ is the Area of one End of the proposed Cylinder; and $1728 \times 30 = 51840$ is its Solidity, and as $907,9224 : 1$ Inch $:: 51840 : 57,097+$, Inches, the Length required.

76. C B represents the Breadth of a River, and B is an Object on the further Side of it, to which we cannot approach nearer than the Side C; but the Land on this Side being level and convenient for taking Stations, from which the Breadth of the River may be readily found, and is here required?

From any Point E, make E I = E C meet B C produced to A, and take I D = D C, then E D will be perpendicular to A B; make A H equal, and parallel to D E, join H E, and produce A H till B E L be-

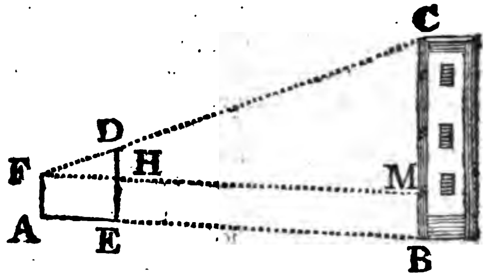


comes a right Line: You may easily perceive (at L) when B, E, and L are all situate in a right Line, by setting up a straight Stick perpendicularly at E, and another at L, the right Lines A C B, and A H L, are formed by setting up a Stick in the same Manner at each of the Points A, C, and H. This premised, let us now suppose that by measuring the several Distances on the Land, we find $AC = 128$, $AL = 100$, $HE = 30$, and $HL = 10$; then in the Right-angled Triangles A B L, H E L, it being as $HL : HE :: AL : AB$, we have $10 : 30 :: 100 : 300 = AB$, and $300 - 128$ $(= AB - AC) = 172 = CB$, the Breadth of the River sought.

77. To find the Altitude of a Tower, Steeple, &c. when its Distance is inaccessible, admitting that the Foot of the Tower and Places of Observation are parallel to the same Horizon?

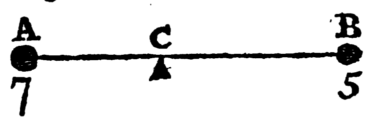
Let

Let B C be the Tower whose Height is sought, and suppose we cannot go any nearer to it than the Place E; find the Distance E B as in the last Problem, which suppose to be 8700 Feet, and let there be placed perpendicularly in the Ground a long Stick E D, likewise a shorter one A F, so as the Observator at A may see the Top of the Tower at C, over the Ends of the two Sticks A F and E D: Let the shorter Stick A F = 5 Feet, and let the Excess of the longer Stick above the shorter, viz. H D = 4 Feet, also let the Distance between those Sticks, viz. A E = F H = 100 Feet, let F M be parallel to the Horizon AB, then will AF (= E H) = 5 Feet = B M; and F M (= A B) = A E + E B = 100 + 8700 = 8800 Feet; and in the similar Triangles F H D, F M C, it will be, as F H : H D :: F M : M C, that is, as 100 : 4 :: 8800 : 352 Feet, to which add 5 Feet (equal to A F = B M), and you will have 357 Feet, or 119 Yards, for the Height of the Tower as required.



78. There is a certain cylindrical Rod 36 Inches long, at the Extremities of which are suspended two Weights, the one of five Pounds the other of seven: I demand the Point of the Rod where these two Weights will be in Equilibrio?

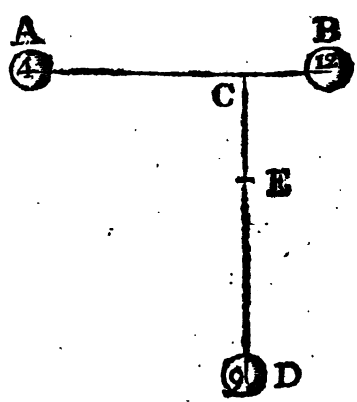
Note, As the Sum of the two Weights is to the Length of the Rod, so is either of the Weights to the Distance from its correspondent Weight to the Point of Balance; thus, as A + B : A B :: B : A C, that is, as 7 + 5, or 12 : 36 :: 5 : 15 Inches equal to A C, the Distance from A the 7 Pound Weight to C, the Point of Balance of these



these two Weights. But if it were required to find the common Centre of Gravity of the Rod and the two Weights, then the Point of Balance will be somewhat different from that above found, and may be easily obtained: For instance, suppose the Rod weighs one Pound; then it is evident that the Momentum of the Rod alone is equal to the Force of one Pound at the Centre: Now half the Length of the Rod (the Distance from the Centre to its End) multiplied into its whole Weight 1 Pound, added to its Length, multiplied into the 5 Pound Weight is $18 \times 1 + 36 \times 5$. or 198, this divided by 13, the Sum of the two Weights and the Weight of the Rod, gives $15 \frac{3}{13}$ Inches, for the Distance from the 7 Pound Weight to the common Centre of Gravity, and is the Point on which the Rod will be kept in Equilibrio, when its Weight is taken into the Account, as it ought: The first Method shews only how to find the common Centre of Gravity of Bodies when their Centres are joined by right Lines, denoting the Distances from each other, as in the following Problem.

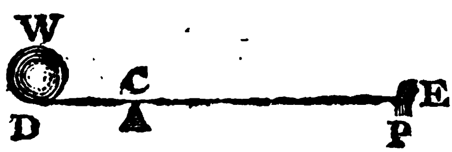
79. A, B, and D, are three Bodies weighing 4, 12, and 9 Pounds respectively; the right Line which joins the Centres of A and B is 40 Inches long, and the Distance between the Centre of Gravity of A and B, and the Centre of the Body D is 50 Inches: Required the common Centre of Gravity of these three Bodies?

First, Find the Centre of Gravity of A and B as in the last Problem, thus, as A + B : A B : : A : C B, that is, as 16 : 40 : : 4 : 10 equal to C B, and the Point C is the Centre of Gravity of A and B; Now imagine that the Weight of the 2 Bodies A and B, viz. 16 lbs. is accumulated



lated into the Point C; and say, as A + B + D : C D : : D : C E, that is, as 25 lbs. (the Weight of the three Bodies) : 50 : : 9 lbs. : 18 Inches, equal to C E, and the Point E is the common Centre of all their Gravitation. In this Manner you find the common Centre of Gravity for any Number or System of Planets. See Martin's Philosophical Grammar, Page 95, second Edition.

80. What Weight will a Person be able to keep in Equilibrio who presses with a Force of 120 lbs. on the End of an equipoised Leaver 60 Inches long, which is to meet with a convenient Fulcrum or Prop (on which it is movable) exactly 12 Inches above the other End of the Machine?

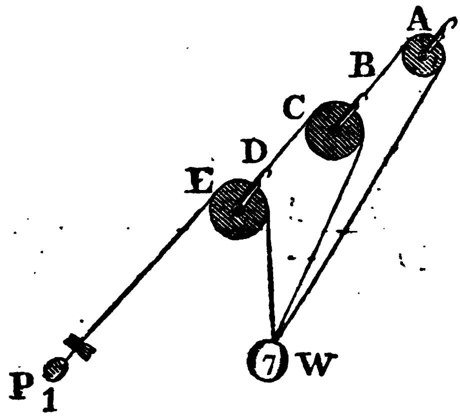


Let P = 120 lbs. the Power or Force with which the Hand presses on the End of the Leaver at E; let D E (= 60 Inches) represent the Leaver movable on the Fulcrum at C; then will D C = 12 Inches, and C E = 48 Inches, and let the Weight sought be denoted by W, then if the Leaver be moved, the Distance D E will represent the Celerity of the Weight W, and C E that of the Power P, but the Weight multiplied into its Celerity is always equal to the Momentum; therefore, the Power P multiplied into C E, its Distance from the Centre of Gravity of Leaver, must be equal to W multiplied into D C, its Distance from the same Point, for the Force on each Side of the Fulcrum C must be equal, otherwise the Equilibrio would be destroyed; therefore it will be reciprocally as C E : P : : D C : W; that is, as 48 : 120 lbs. : : 12 : 489 Pounds, the Weight required.

So that if the Hand presses on the End of the Leaver with a Force any Thing greater than that of 120 lbs. it will raise the Weight W of 480 Pounds, placed as here supposed: And though there be Leavers of several Sorts,

Sorts, yet what is here said is equally applicable to them all.

81. A C E is a Tackle of Pulleys, whereof the upper one A is fixed, and each has a Rope fixed to the Weight W, consisting of 28 cwt. Now admitting E and C to rise and descend freely with the Weight W, it is required to find what Power applied to the Rope at P, will sustain the Weight W?



Here it is obvious, that when the Weight W is raised one Inch, the Rope A B will be lengthened as much; and so the Pulley C will descend one Inch, by which Means the Rope C D will be lengthened two Inches, and one by the rising of the Weight W; wherefore the Pulley E will descend three Inches; and thus the Rope E P will be lengthened six Inches (namely three on each Side) also, the Rising of the Weight will cause it to lengthen one Inch more, so that the Power P goes through seven Inches while the Weight W rises one, hence it will be reciprocally as 1 : W : : 7 : P, that is, as 1 : 28 cwt. : : 7 : 4 cwt. the Force at P required.

82. An Annuity of 20l. per Annum is forborn, or unpaid 7 Years, what will then be due at 6 per Cent. compound Interest?

First,

First, I add 1l. to (1,06) one Pound and its Interest for one Year, and the Sum is 2,06 the second Year's Amount, this multiplied by 1,06, and the Product added to 1l. gives 3,1836 for the third Year's Amount, which being likewise multiplied by 1,06, and the Product added to 1l. gives 4,374616 for the fourth Year's Amount, and by proceeding on in this Manner you will find the seventh Year's Amount of 1l. to be 8.393837649856, which being multiplied by 20l. the Annuity, gives 167,87675l. + or 167l. 17s. 6 ¼d. the Answer.

83. What Annuity, to continue seven Years, may be purchased for 120l. 5s. at 6 per Cent. compound Interest?

First, $1 + 1,06 + \overline{1,06}^2 + \overline{1,06}^3 = 4,374616$, the Amount of 1 Pound for 4 Years, and its Amount for 7 Years, is 8,393837649856, which, being divided by $\overline{1,06}^7$, or 1,50363025 +, gives 5,5823814 +, the present Worth of 1l. for 7 Years; and as 5,5823814l. : 1l. : : 120, 25l. : 21,5098l. or 21l. 10s. 9 ¼d. the Annuity required.

By this Operation you may compose a Table exhibiting the present Worths of 1l. for any Rate and Time; and by the Solution to Question 82, you may construct a Table shewing the Amount of 1 Pound for any Time and Rate proposed; but if Unity or 1 be divided by the present Worth of 1l. for any Time and Rate of Interest, the Quotient will be the Annuity which 1l. will purchase for that Time, at the Rate assigned; whence also a Table may be constructed which shall readily produce the Annuity required, by multiplying the Sum, with which it is to be purchased, by the Number in the Table corresponding to the given Time and Rate of Interest. For Instance, in the last Question, if Unity be divided by 5,5823814, the Quotient will be 0,179135, which, being multiplied by 120,25, gives 21,5098, or 21l. 10s. 9 ¼d. for the Annuity, the very same as before.

I shall solve the next two following Questions by double Position; but besides the common Way of multiplying each Supposition by the Error of the other supposed Number, there is another Method used sometimes by Alge-

braists, in solving Equations, containing Surds, and exponential Quantities, which Method I shall make Use of here. And the Rule is this: Having taken two convenient Numbers for your Suppositions, work with both according to the Nature of the Question, and having found the Errors of their respective Results; say, as the Sum of the Errors, if unlike, or their Difference, if alike (that is, both Excesses or both Defects) is to the Difference of the supposed Numbers, so is the least Error to a fourth Number, which is a Correction of the Number belonging to the least Error, and is to be added to, or subtracted from it, according as the Result of that Number is too little or too great. In like Manner you try this Number and the nearest of the former, or else take a new supposed Number, then find their Errors as before, and you will get a Number still nearer; and thus by repeating the Operation you may continually approximate as near as you will to the true Number sought.

84. If an Annuity of 50l. forborne 18 Years, amount to 1342,75l. What Rate of Interest was allowed?

First, find the Amount of 1l. for 18 Years, at any assumed Rate of Interest; this may be very speedily done by Logarithms, thus, suppose the Interest to be 5 per Cent. then the Log. of 1,05 the Ratio (that is, the Amount of 1l. and its Interest for one Year) is 0,0211893, which, being multiplied by 17, the Product is 0,3602181, and is the Log. of 2,292 +, the last Term in a Series of Numbers, increasing in Geometrical Progression; from which take (1l.) the first Term, divide the Remainder by the Ratio, less one, to the Quotient add the last Term, and you will have $2{,}292 - 1 \div 1{,}05 - 1 + 2{,}292 = 28{,}132$, the Sum of all the Terms, or the Amount of 1l. for 18 Years, and consequently $28{,}132 \times 50 = 1406{,}6$l. is the Amount of 50l. for the same Time, which is too much by ($1406{,}6 - 1342{,}75$, or, 63,85, wherefore I make a Supposition at 4 per Cent. and working as before, find the Amount to be 1282,25l. which is too little by 60,5, whence as 124,35 the Sum of the Errors is to 1 the Difference of the supposed Rates, so is 60,5, the least Error, to ,48 +, which added to 4, the Rate belonging to the least

least Error, gives 4,48 for the assumed Rates of Interest once corrected, and seeing that the Interest comes out near $4\frac{1}{2}$ per Cent. I therefore make Trial with $4\frac{1}{2}$, and find it to succeed; for the Amount of 1l. for 18 Years at $4\frac{1}{2}$ per Cent. is 26,855, which, being multiplied by 50, the Product is 1342,75l. the given Amount; therefore $4\frac{1}{2}$ per Cent. is the Rate of Interest required.

85. Suppose a Bookseller purchases a Work for 40l. and pays for Printing 1000 Copies thereof 15l. for Paper 20l. and for Advertising and other incident Charges 10l. Now if he sells the Edition at 3s. each Copy in 10 Years (that is, 100 Copies every Year) what does he gain per Cent?

Here the Bookseller lays out 85l. to purchase an Annuity of 15l. per Annum, to continue 10 Years, and consequently the Interest is pretty high, therefore I assume it at 14 per Cent. and find what Annuity 85l. will purchase for 10 Years at this Rate of Interest.

Thus, the Log. of 1,14 the Ratio is ,0569048, this multiplied by 10, the Product 0,569048, is the Log. of $\overline{1,14}^{10}$, or 3,707 +.

Hence by a Theorem in Geometrical Progression (demonstrated in the Algebraic Part of this Work) we have $\frac{3,707-1}{1,14-1}=\frac{2,707}{0,14}=19,335+$, the Amount of 1l. for 10 Years; and as $\overline{1,14}^{10}$, that is, as 3,707 : 1 : : 19,335 : 5,2 the present Worth of 1l. for the same Time; and by Problem 83, it will be as 5,2 : 1 : : 85 : 16,346, the Annuity which 85l. will purchase to continue 10 Years at 14 per Cent, which is too much by 16,346—15, or 1,346; wherefore I assume the Rate at 12 per Cent. and proceeding as above, find the Annuity which 85l. will purchase at 12 per Cent. for 10 Years, to be 15,039, which is too much by ,039; hence, as 1,307 the Difference of the Errors is to 2, the Difference of the assumed Rates, so is ,039, the least Error, to ,05, which being taken from 12, the Rate from which the least Error arose, leaves 11,95, or 11l. 19s. for the Gain per Cent. required.

 From

and it will be found upon Trial, that the Interest here brought out is sufficiently near the Truth.

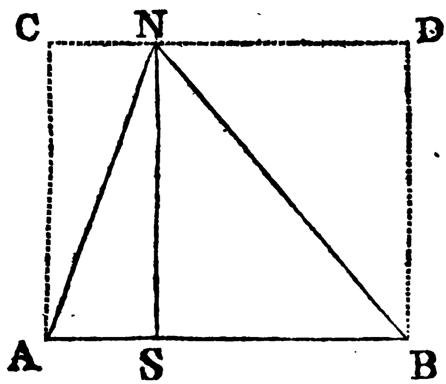
86. A Bond was made on the first of May 1771, at 6 per Cent. per Annum for the Sum of 1500l. On the 13th of July 1777, 1018l. was paid off, and a fresh Bond entered into for the Remainder at 5 per Cent. per Annum. At the Time the Interest of this last was 156l. there was paid off 896l. The old Bond being then taken up, a new one was given for the Residue, which being paid off on the 16th of February, 1782, the Bond-Owner took no more than 322l. 16s. in full Payment: At what Rate then did he take Interest per Cent. per Annum, upon the last Renewal of the Bond?

From the first of May 1771, to July 13, 1777, are 6 Years and 73 Days, equal to 6,2 Years, and, as 100l. $\times$ 1 Year : 6l. : : 1500l. $\times$ 6,2 Years : 558l. the Interest of 1500l. for 6,2 Years; therefore $1500 + 558 - 1018 = 1040$l. which remained due to the Owner of the Bond on July 13th, 1777, and as 100l. —— 1 Year —— 5l.
1040 —— * —— 156
hence $156 \times 100 \div 1040 \times 5 = 3$ Years, the Time in which the Interest of 1040l. becomes equal to 156l. so that on July 13th, 1780, there was due $1040 + 156 - 896 = 300$l. but from July 13, 1780, to February 16th, 1782, is 1 Year and 219 Days, equal to 1,6 Year, in which Time 300l. amounted to 322l. 16s. or to 322,8l. therefore $322,8 - 300 = 22,8$l. is the Interest of 300l. for 1,6 Year, and as $300 \times 1,6$: 22,8l. : : 100×1 : 4,75l. or 4l. 15s. the Rate per Cent. required.

87. Two Ships failed from two Ports in the Latitude of 51° 25′ North, the Easternmost between the North and West, the other between the North and East, till they both met in the Latitude of 58° 30′ North, when the Angle made with each others Courses was 74°, and upon comparing their Reckonings together, found that the Ratio of the Easternmost Ship's Distance was to that of the Westernmost, as 5 to 3; Required each Ship's Course, Distance

From

From these two Operations it appears, how very easy Numbers may be involved to high Powers by Logarithms, sailed, and Departure, together with the Distance of the two Ports.



Draw the Right-lines A N, B N, containing the given Angle A NB $=74^\circ$; then will N denote the Place where the Ships met; join A B and let A and B denote the two Ports from which the Ships sailed, then will the Distances sailed be denoted by A N and B N respectively, make N S perpendicular to A B, draw the Meridians A C and B D each parallel and equal to N S; draw the Departures C N and D N: The Courses steered are the Angles C A N and D B N, and in the Triangle A N B, we have S N ($=58^\circ\ 30'-51^\circ\ 25'=7^\circ\ 5'$, and $7^\circ\ 5' \times 60=425$ Miles, the Difference of Latitude; and the Sum of the Angles at the Base $=180^\circ-74^\circ=106^\circ$, hence it will be as 8 the Sum of the Numbers representing the Ratio of the Sides is to 2 their Difference, so is the Tangent of 53°, half the Sum of the Angles at the Base, to the Tangent of half their Difference; that is,

As the Logarithm of 8, which is		0,9030900
Is to the Log. of 2		0,3010300
So is the Tangent of 53°		10,1228856
To the Tangent of	$18^\circ\ 21'$	9,5208256

Then

Then the Angle N A S $= 53° + 18° 21' = 71° 21'$ whose Complement is $18° 39'$, and is equal to the (Angle CAN, the) Westernmost Ship's Course corresponding to N. by E. $\frac{3}{4}$ E fere; and the Angle S B N $= 53° - 18° 21' = 34° 39'$, whose Complement $55° 21'$ is equal to the (Angle N B D, the) Easternmost Ship's Course, which answers to N W by W nearly.

Now to find the Distances and Departures, say, as the Sine of the Angle N A S ($71° 21'$) is to S N (425 Miles) the Difference of Latitude, so is Radius ($90°$) to 448,5 Miles, equal to A N; then $448{,}5 \times \frac{5}{3} = 747{,}5$ Miles $=$ B N; and as Radius is to A N, so is the Sine of the Angle C A N ($18° 39'$) to 143,424 Miles $=$ C N; lastly, as Radius is to B N, so is the Sine of the Angle N B D to 614,923 Miles equal to D N, whence C N $+$ N D $= 143{,}424 + 614{,}923 = 758{,}347$ Miles $=$ C D $=$ A B, the Distance between the Ports.

88. Four Men have a Sum of Money to be divided among them in such a Manner, that the first shall have $\frac{1}{3}$ of it, the second $\frac{1}{4}$, the third $\frac{1}{6}$, and the fourth the Remainder, which was 28l. what is the Sum?

First, $1 - \frac{1}{3} - \frac{1}{4} - \frac{1}{6} = \frac{1}{4}$, the Remainder, which, by the Question, is equal to 28l. And as $\frac{1}{4} : 28\text{l.} :: \frac{1}{1} : 112\text{l.}$ the Answer.

89. It is required to find how far the Inhabitants of a certain Village, situate by the Side of the River Orwell, in $52° 8'$ North Latitude, are carried in a Minute's Time, by the Earth's Rotation about its Axis?

First, I find how many Miles will make a Degree of Longitude in the Latitude proposed, thus,

As Radius (1) is to ,613826 the Co-sine of $52° 8'$, so is 69,5 Miles, a Degree of Longitude on the Equator, to 42,660907 Miles, the Length of a Degree of Longitude in the Latitude of $52° 8'$, which being divided by 4, because a Degree of Longitude is described in 4 Minutes, gives 10,66522675 Miles for the Velocity, or Space, through which the Village passes in each Minute of Time.

And by proceeding as above, you may construct a Table,

ble, ſhewing how many Miles will make a Degree of Longitude, at every Degree and Minute of Latitude.

In this Solution I have uſed natural Sines to the Radius 1; and the ſame may be performed by artificial Sines; but in the Application of Algebra to Trigonometry, we generally uſe the natural Sines and Tangents, and they muſt be actually multiplied, divided, &c. (as Occaſion requires) by the Rules of Decimal Arithmetic.

So that the above Operation may alſo ſerve as a Preliminary to the Uſe of the Table of Natural Sines, &c.

In this Operation I have reckoned 69 $\frac{1}{2}$ Miles to a Degree on the Equator, ſo that the Anſwer is in Engliſh Miles, but if 60 Miles had been taken for a Degree, then the Anſwer would have been Geographical Miles, and the Solution would have been more ſimple.

Some Authors reckon 69 $\frac{4}{11}$ Engliſh Miles to a Degree on the Equator.

SECTION I.

OF NOTATION.

ALGEBRA is a Kind of ſpecious Arithmetic, or an Arithmetic in Letters, and it is a Science which ſhews the Compariſon of abſtract Quantities in a general Manner: It properly follows Arithmetic and Geometry, but is vaſtly ſuperior in Nature to both, as it can ſolve Queſtions quite beyond the Reach of either of them.

In Algebra, the given or known Quantities are uſually denoted by the firſt Letters of the Alphabet, as, a, b, c, d, &c. and the unknown, or required ones, by the laſt Letters, as x, y, and z.

There are, moreover, in Algebra, certain Signs or Notes made uſe of, to ſhew the Relation and Dependence of Quantities one upon another, whoſe Signification the Learner ought firſt of all, to be made acquainted with.

The Sign +, ſignifies Addition, and ſhews that the Numbers or Quantities between which it is placed are added together in one Sum. Thus 10 + 6 expreſſes the

Sum of 10 and 6 added together, which make 16. This Sign (+) is usually expressed by the Word Plus (or more) thus $a + b$ is read a plus b and shews that the Number represented by b is added to that represented by a, and expresseth the Sum of a and b; so that if a were 10, and b 6, then would $a + b$ be $10 + 6$, or 16, as above.

In like Manner $a + b + c$ denotes the Number arising by adding all the three Quantities a, b, and c together; so that if a were 10, b 6, and c 4, then would $a + b + c$, be $10 + 6 + 4$, or 20. For 10 plus 6 plus 4 (or 10, 6 and 4 being added together) make 20.

The Sign —, signifies Subtraction, and shews that the Number or Quantity which comes after it is subtracted from that which stands before it, and it stands likewise for the Word Minus, or less; thus $10-6$ is read 10 minus 6, which is 4; for 6 being subtracted from 10 there remains 4. In like Manner $a-b$ is read a minus b, and it expresses the Difference of the two Quantities a and b; so that if a were 10 and b 6, then would $a-b$ be $10-6$ or 4, as before. Moreover, $a + b, - c$ shews that the Number represented by c is subtracted from that represented by the Sum of a and b, thus if a be 10, b 6, and c 4, then will $a + b - c$ be $10 + 6 - 4$, or 12. For 6 added to 10 make 16, from which 4 being subtracted there remains 12.

Note. Those Quantities before which the Sign + is placed are called positive, or affirmative, and those before which the Sign — is placed, negative.

And it is to be observed that the Sign of a negative Quantity is never omitted, nor the Sign of an affirmative one, except it be a single Quantity, or the first in a series of Quantities, then the Sign + is frequently omitted, thus a signifies the same as $+ a$, and the Series $a + b - c + d$ the same as $+ a + b - c + d$; so that if any single Quantity, or if the first Term in any Number of Terms has not a Sign before it, then it is always understood to be affirmative.

If an Algebraical Quantity consists of two Terms, it is called a Binomial, as $a + b$; if of three Terms, a Trinomial, as $a + b + c$; and if there be more Terms it is called a Multinomial; all which are compound Quantities.

Single,

Single, or ſimple Quantities, conſiſt of one Term only, as a, b, x, y. *

The Sign $\times$, ſignifies that the Quantities between which it ſtands are multiplied together. Thus $a \times b$ denotes the Product of the two Quantities a and b; ſo that if a were 10 and b 6, then would $a \times b$ be 10×6, or 60. For 6 times 10 is 60: Again, $a \times b \times c$ expreſſes the Product ariſing by multiplying the Quantities a, b, and c, continually together, ſo that if a be 2, b 3, and c 4, then will $a \times b \times c$ be $2 \times 3 \times 4$, or 24, for three Times 2 is 6 and 4 times 6 is 24.

But in multiplying ſimple Quantities we frequently omit the Sign $\times$, and join the Letters; thus ab ſignifies the ſame as $a \times b$; and abc the ſame as $a \times b \times c$.

*And theſe Products, viz. $a \times b$ or ab, and abc, are called ſingle, or ſimple Quantities, as well as the Factors (viz. a, b, c.) from which they were produced, and the ſame is to be obſerved of the Products ariſing from the Multiplication of any Number of ſimple Quantities. When a compound Quantity is to be expreſſed as multiplied by a ſimple one, then we place a Sign of Multiplication between them, and draw a Line over the compound Quantity only; and when compound Quantities are to be repreſented as multiplied together, then we draw a Line over each of them, and connect them with a proper Sign. Firſt $\overline{a+b} \times c$ denotes that the compound Quantity $a+b$ is multiplied by the ſimple Quantity c; (ſo that if a were 10, b 6, and c 4, then would $\overline{a+b} \times c$ be $\overline{10+6} \times 4$, or 16×4, which is 64) and $\overline{a+b} \times \overline{c+d}$ expreſſes the Product of the compound Quantities $a+b$, and $c+d$ multiplied together.

The Note . (or a full Point) and the word into, are likewiſe uſed as Signs of Multiplication inſtead of $\times$: Thus $\overline{a+b} \,.\, \overline{c+d}$ and $\overline{a+b}$ into $\overline{c+d}$ both ſignify the ſame Thing as $\overline{a+b} \times \overline{c+d}$.

The Sign $=$, is called the Sign of Equality, and is uſed to ſignify that the Quantities on each Side of it are equal to one another; Thus $6+4=10$, denotes that

6 plus

6 plus 4 is equal to 10; and $x = a + b$ shews that x is equal to the Sum of a and b.

When we would express that one Quantity as a, is greater than another, as b, we write $a \sqsubset b$, or $a > b$; and if we would express Algebraically that a is less than b, we write $a \sqsupset b$, or $a < b$.

When we would express the Difference between two Quantities (as a and b) while it is unknown which is the greater of the two, we write thus $a \backsim b$, which denotes the Difference of a and b indefinitely, whichever of the two is greatest.

The Note $\therefore$ or $\because$ is used by Algebraists to signify the Word *ergo*, or therefore.

Powers of the same Quantities or Factors, are derived from the Products of their Multiplication: Thus $a \times a$, or aa, denotes the Square or second Power of the Quantity represented by a, $a \times a \times a$, or aaa, expresses the Cube, or third Power, and $a \times a \times a \times a$, or $aaaa$, denotes the Biquadrate or fourth Power of a, &c.

And it is to be observed that the Quantity a is the Root of all these Powers. Suppose $a = 5$, then will aa ($= a \times a = 5 \times 5$) $= 25$, the Square of 5 (a), aaa ($= a \times a \times a = 5 \times 5 \times 5$)$=125$, the Cube of 5 ($a$), and $aaaa$ ($=a \times a \times a \times = 5 \times 5 \times 5 \times 5$) $= 625$, the fourth Power of 5 (a).

Powers are likewise represented, by placing above the Root to the right Hand, a Figure expressing the Number of Factors that produce them.

Thus, instead of aa we write a^2, instead of aaa we write a^3, instead of $aaaa$, we write a^4, &c.

These Figures, which express the Number of Factors that produce Powers, are called their Indices or Exponents; thus 2 is the Index or Exponent of a^2; 3 is that of x^3; 4 is that of x^4, &c. But the Exponent of the first Power, though generally omitted, is unity or 1, thus a^1 signifies the same as a (namely the first power of a), $a \times a$ the same as $a^1 \times a^1$, or a^{1+1}, that is, a^2, and $a^2 \times a$ is the same as $a^2 \times a^1$, or a^{2+1}, or a^3; hence Powers of the same Quantity or Root, may be always multiplied by merely

merely adding their Exponents; thus $a^3 \times a = a^3 \times a^1 = a^{3+1} = a^4$, $a^3 \times a^2 = a^{3+2} = a^5$ (the fifth Power of a), and $x^4 \times x^3 = x^{4+3} = x^7$ (the seventh Power of x.)

In expressing Powers of compound Quantities we usually draw a Line over the given Quantity, and at the End of the Line place the Exponent of the Power: Thus $\overline{a+b}|^2$ denotes the Square or second Power of $a + b$, considered as one Quantity, $\overline{a+b}|^3$ the third Power, $\overline{a+b}|^4$ the fourth Power, &c. and it may be observed that the Quantity $a + b$ (called the first Power of $a + b$) is the Root of all these Powers.

Let $a = 4$, and $b = 2$, then will $a + b$, become $4 + 2$, or 6, and $\overline{a+b}|^2 = \overline{4+2}|^2 = 6^2 = 6 \times 6 = 36$, the Square of 6 $(a + b)$ also $\overline{a+b}|^3 = \overline{4+2}|^3 = 6^3 = 6 \times 6 \times 6 = 216$, the Cube of 6 $(a + b)$.

Powers of the same Root of compound Quantities are likewise multiplied by adding their Exponents.

Thus, $\overline{a+b}|^4 \times \overline{a+b} = \overline{a+b}|^4 \times \overline{a+b}|^1 = \overline{a+b}|^{4+1} = \overline{a+b}|^5$; $\overline{a+b}|^4 \times \overline{a+b}|^2 = \overline{a+b}|^{4+2} = \overline{a+b}|^6$ and $\overline{x+a}|^5 \times \overline{x+a}|^3 = \overline{x+a}|^{5+3} = \overline{x+a}|^8$.

But if the Quantities which are to be expressed as multiplied together, be Powers of different Roots, their Exponents must not be added; thus $a^2 \times x^3 = a^2 x^3$, $a^2 x^3 \times a^3 x^4 = a^5 x^7$, and $\overline{a+b}|^4 \times \overline{c+d}|^3$ denotes the Product arising by multiplying the fourth Power of $a + b$, by the third Power of $c + d$.

The Sign $\div$ is used to signify that the Quantity which stands before it is divided by the Quantity which comes after it: thus $c \div b$ denotes that c is to be divided by b; so that if c be 20, and b 5, then will $c \div b$ be $20 \div 5$, or 4; for 20 being divided by 5 the Quotient is 4; thus likewise $\overline{a+b} \div \overline{c-d}$ shews that $a + b$ is to be divided by $c - d$.

Also the Mark) is sometimes used as a Note of Division; thus $a + b)\ a\,b$, denotes that the Quantity $a\,b$ is to be divided by the Quantity $a + b$, and so of others.

But

But the Division of Algebraic Quantities is most commonly expressed, by writing down the Divisor under the Dividend with a Line between them (in the Manner of a vulgar Fraction) thus $\frac{a}{c}$ represents the Quantity arising by dividing a by c; so that if a be 144, and c 4, then will $\frac{a}{c}$ be $\frac{144}{4}$ or 36, for 144 being divided by 4, the Quotient is 36; and $\frac{a+b}{a-c}$ denotes the Quantity arising by dividing $a+b$, by $a-c$, suppose $a=12$, $b=6$, and $c=9$, then will $\frac{a+b}{a-c}$ become $\frac{12+6}{12-9}$ or $\frac{18}{3}=6$.

These literal Expressions, namely $\frac{a}{c}$, and $\frac{a+b}{a-c}$, are called Algebraic Fractions; whereof the upper Parts are called the Numerators, and the lower the Denominators; thus a is the Numerator of the Fraction $\frac{a}{c}$, and c is its Denominator; $a+b$ is the Numerator of $\frac{a+b}{a-c}$, and $a-c$ is its Denominator.

The Sign $\sqrt{\ }$, is called a radical Sign, and is used to express the square Root of the Quantity to which it is prefixed; thus $\sqrt{16}$ signifies the square Root of 16, which is 4. In like Manner $\sqrt{ab}$ expresses the square Root of ab, and $\sqrt{\frac{a+b}{c}}$ denotes the square root of $\frac{a+b}{c}$; suppose $a=100$, $b=44$ and $c=4$, then will $\sqrt{\frac{a+b}{c}}$ be $\sqrt{\frac{100+44}{4}}$ or $\sqrt{\frac{144}{4}}$ that is, $\sqrt{36}=6$ but $\frac{\sqrt{a+b}}{c}$ because the Line which separates the Numerator from the Denominator is drawn below the Sign $\sqrt{\ }$, signifies that the square Root of $a+b$ is to be divided by c. Let $a=100$, $b=44$ and

and $c = 4$ (as before) then will $\frac{\sqrt{a+b}}{c}$ become $\frac{\sqrt{100+44}}{4}$ or $\frac{\sqrt{144}}{4} = \frac{12}{4} = 3$. For the ſquare Root of 144 is 12, which being divided by 4, the Quotient is 3.

Moreover, $\sqrt{a+b+\sqrt{bc}}$, denotes that the ſquare Root of bc is added to $a+b$, and that the ſquare Root of the Sum is extracted; ſuppoſe $a = 19$, $b = 2$, and $c = 8$, then will $\sqrt{a+b+\sqrt{bc}}$, become $\sqrt{19+2+\sqrt{2\times 8}}$, or $\sqrt{21+\sqrt{16}}$, or $\sqrt{21+4}$, that is, $\sqrt{25} = 5$. The ſame Sign $\sqrt{}$ with a Figure over it, is alſo uſed to expreſs the Cube, or Biquadratic Root, &c. of any Quantity; thus $\sqrt[3]{8}$ expreſſes the Cube Root of 8, which is 2, and $\sqrt[4]{81}$ denotes the Biquadratic. Root of 81, which is 3. In the ſame Manner $\sqrt[3]{ab}$, denotes the Cube Root of ab, and $\sqrt[4]{ab+bc}$ repreſents the Biquadratic Root of $ab+bc$; and ſo of others. Quantities thus expreſſed are called radical Quantities, or Surds; whereof thoſe, conſiſting of one Term only, as $\sqrt{a}$ and $\sqrt{ax}$ are called ſimple Surds; and thoſe conſiſting of ſeveral Terms, as $\sqrt{ab+cd}$ and $\sqrt[4]{a^2-b^2+bc}$, are compound Surds.

When any Quantity is to be taken more than once, the Number is to be prefixed, which ſhews how many Times it is to be taken, and the Number ſo prefixed is called the Numeral-coefficient; thus $2a$ ſignifies twice a, or a taken twice, and the Numeral-coefficient is 2; $3x^2$ ſignifies that the Quantity x^2 is multiplied by 3, and the Numeral-coefficient (of $3x^2$) is 3, alſo $5\sqrt{x^2+a^2}$ denotes that the Quantity $\sqrt{x^2+a^2}$ is multiplied by 5, or taken 5 Times.

When no Number is prefixed, an Unit or 1 is always underſtood to be the Co-efficient: thus 1 is the Co-efficient of

of a or of x; for a signifies the same as $1\,a$, and x the same as $1\,x$; since any Quantity multiplied by Unity is still the same.

Moreover, if a and d are given Quantities, and x^2 and y required ones, then $a\,x^2$ denotes that x^2 is to be taken a Times, or as many Times as there are Units, &c. in a, and $d\,y$ shews that y is to be taken d times; so that the Co-efficient of $a\,x^2$ is a, and that of $d\,y$ is d; suppose $a = 6$, and $d = 4$, then will $a\,x^2 = 6\,x^2$, and $d\,y = 4\,y$. Again, $\frac{1}{2}\,x$, or $\frac{1\,x}{2}$ denotes the half of the Quantity x, and the Co-efficient of $\frac{1}{2}\,x$ is $\frac{1}{2}$, so likewise $\frac{3}{4}\,x$, or $\frac{3\,x}{2}$ signifies $\frac{3}{4}$ of x, and the Co-efficient of $\frac{3}{4}\,x$ is $\frac{3}{4}$. Quantities are said to be like that are represented by the same Letters under the same Powers, or which differ only in their Co-efficients; thus $3\,a$, $5\,a$, and a are like Quantities, and the same is to be understood of the Radicals $\sqrt{x^2 + a^2}$ and $7\sqrt{x^2 + a^2}$. But unlike Quantities are those which are expressed by different Letters, or by the same Letters under different Powers: thus $2\,a\,b$, a^2b, $2\,a\,b\,c$, $5\,a\,b^2$, $4\,x^2$, y, y^2, and z^2 are all unlike Quantities.

The Note : being placed between Numbers or Quantities, signifies the Word to; and :: signifies the Words so is, and shews that the Quantities on each side of it are proportional; thus $a : b :: c : d$, denotes that a is in the same Proportion to b, as c is to d, and is thus read, as a is to b, so is c to d. Suppose $a = 2$, $b = 4$, and $c = 3$, then will $d = 6$; for as $2 : 4 :: 3 : 6$. Here multiplying the second and third Numbers together, the Product is 4×3, or 12, which being divided by 2, the first Number, gives 6 for d, the fourth Number, and it is obvious that 2 is to 4, as 3 is to 6; for 2 is the half of 4, and 3 is the half of 6.

The double Sign $\pm$ signifies plus, or minus, the Quantity which immediately follows it, and being placed between two Quantities it denotes their Sum, or Difference: thus $\frac{1}{2}\,a \pm \sqrt{\frac{a^2}{4} - b}$, shews that the Quantity $\sqrt{\frac{a^2}{4} - b}$ is

is to be added to, or subtracted from $\frac{1}{2}a$: This ambiguous Sign is often used in solving Problems, producing adfected quadratic Equations, in which Case sometimes the Sum, sometimes the Difference, and sometimes both the Sum and Difference of the two Quantities between which it is placed are to be taken alternately, all which must be determined from the Nature and Conditions of the Problem.

Of Quantities expressed by general Exponents.

By a general Exponent, I mean one that is denoted by a Letter instead of a Figure; thus the Quantity x^m has a general Exponent (viz. m), and universally denotes the m^{th} Power of the Root x; Suppose $m = 2$, then will $x^m = x^2$, if $m = 3$, then will $x^m = x^3$, if $m = 4$, then $x^m = x^4$, &c.

In like Manner $\overline{a-b}^m$ expresses the m^{th} Power of $a-b$.

This Root (viz. $a-b$) is called a residual Root, because its Value is no more than the Residue, Remainder, or Difference between its Terms a and b: It is likewise called a binomial (as well as $a+b$) because it is composed of two Parts connected together by the Sign $-$.

As x^m signifies the m^{th} Power of x, so x^{2m} denotes the Square of x^m, x^{3m} expresses the Cube of x^m, &c. be the Exponent m what it will. Suppose $m = 1$, then will $x^m = x^1$ $(=x)$ and $x^{2m} = x^{2\times1} = x^2$, $x^{3m} = x^{3\times1} = x^3$; $x^{4\times m} = x^{4\times1} = x^4$, &c. so that the second, third, fourth, &c. Powers of any Quantity may be obtained by multiplying its Exponent by 2, 3, 4, &c. respectively; and consequently the Square, Cube, Biquadratic, &c. Roots of any Quantity may be represented by dividing its Exponent severally by 2, 3, 4, &c, &c. Thus $x^{\frac{m}{2}}$ denotes the square Root of x^m (be m what it will): Suppose $m = 1$, as before, then will $x^{\frac{m}{2}} = x^{\frac{1}{2}}$ the square Root of $(x^1 =)\ x$; $x^{\frac{m}{3}} = x^{\frac{1}{3}}$ the Cube Root of $(x^m = x^1 =)\ x$, $x^{\frac{m}{4}} = x^{\frac{1}{4}}$, the Biquadratic Root

Root of ($x^m =$) x; $x^{\frac{m}{5}} = x^{\frac{1}{5}}$ the surſolid Root of (x^m =) $x^1 = x$, &c. Hence the Method of expreſſing the Root of a Quantity by a vulgar Fraction will naturally follow: Thus $a^{\frac{1}{2}}$ ſignifies the ſame thing with $\sqrt{a}$; and $\overline{a + ab}|^{\frac{1}{3}}$ the ſame as $\sqrt[3]{a + ab}$; likewiſe $a^{\frac{2}{3}}$ denotes the Square of the Cube Root of the Quantity a: Suppoſe $a = 64$, then will $a^{\frac{2}{3}} = 64^{\frac{2}{3}} = 4^2 = 16$. For the Cube Root of 64 is 4, and the Square of 4 is 16. Again $\overline{a + b}|^{\frac{5}{4}}$ expreſſes the fifth Power of the Biquadratic Root of $a + b$; ſuppoſe $a = 9$, and $b = 7$, then will $\overline{a + b}|^{\frac{5}{4}} = \overline{9 + 7}|^{\frac{5}{4}} = \overline{16}|^{\frac{5}{4}} = 2^5 = 32$. For the biquadratic Root of 16 is 2, and the fifth Power of 2 is 32; alſo $a^{\frac{1}{n}}$ ſignifies the n^{th} Root of a; if $n = 4$, then will $a^{\frac{1}{n}} = a^{\frac{1}{4}}$, if $n = 5$, then will $a^{\frac{1}{n}} = a^{\frac{1}{5}}$, &c. Moreover $\overline{a + b}|^{\frac{m}{n}}$ denotes the m^{th} Power of the n^{th} Root of $a + b$; if $m = 3$, and $n = 2$, then will $\overline{a + b}|^{\frac{m}{n}} = \overline{a + b}|^{\frac{3}{2}}$, namely the cube of the ſquare Root of the Quantity $a + b$; and as $a^{\frac{1}{n}}$ equals $\sqrt[n]{a^1}$, or $\sqrt[n]{a}$, ſo $\overline{a + b}|^{\frac{m}{n}} = \sqrt[n]{\overline{a + b}|^m}$, namely the n^{th} Root of the m^{th} Power of $a + b$; ſo that the m^{th} Power of the n^{th} Root, or the n^{th} Root of the m^{th} Power of a Quantity are the very ſame in Effect, though differently expreſſed.

Fractions, by which the Roots of Quantities are denoted, are called Exponents, their Numerators ſhewing the Powers, and their Denominators the reſpective Roots of the Quantities ſo extracted; thus the Exponent of $a^{\frac{1}{2}}$, is $\frac{1}{2}$, that of $a^{\frac{2}{3}}$ is $\frac{2}{3}$, that of $\overline{a + b}|^{\frac{5}{4}}$ is $\frac{5}{4}$, that of $a + b^{\frac{m}{n}}$ is $\frac{m}{n}$, and ſo of others.

Of

Of Exponential Quantities.

An Exponential Quantity is a Power whose Exponent is a variable Quantity; as x^x is called an Exponential Quantity, and when such Quantities occur in the Resolution of Problems, and are either given or found equal to some other Quantities, then such Equations are frequently solved by the Help of Logarithms: It is true I am not going to solve Exponential Equations in this Place; however, I shall give a Definition of these Quantities in the most simple Manner I can, that the young Algebraist may have an abstract Idea of them, and not be surprised nor retarded when he meets with such Quantities afterwards. First, then, the Quantity x^x denotes the x Power of x, for here x is the Exponent of x^x, and x is evidently equal to itself, be it what it will: Suppose $x=2$, then will $x^x=2^2=4$, if $x=3$, then will $x^x=3^3=3\times3\times3=27$, if $x=4$, then wil $x^x=4^4=256$, &c. And here it is to be observed, that x is the Root of the Power x^x.

Now any one who understands the Use of Logarithms, knows that the Logarithm of the Power is produced by multiplying the Logarithm of the Root by its Exponent; and the Logarithm of the Root is had by dividing the Logarithm of the Power by its Exponent: Thus, if $x=5$ then will $x^x=5^5$ $(=5\times5\times5\times5\times5)=3125$, and the Logarithm of the Root 5 (x) viz. ,6989700, being multiplied by the Exponent 5 (x) the Product is $,6989700\times5$ or 3,4948500, and is the Logarithm of (3125) the Number corresponding to the Power x^x (when $x=5$) that is, Log. $,6989700\times5=$ Log. $x\times x=$ Log. of $x^x=$ Log. of $3125=3,4948500$, and consequently if (3,4948500) the Log. of the Power $x^x=3125$, be divided by the Exponent 5, the Quotient will be (,6989700) the Log. of 5, equal to the Root x. Here the Exponent and Root of the Quantity x^x are equal to each other, but the Exponent (y) of the Quantity x^y, may be equal to, or greater, or less than its Root (x). However, if x^y be equal to any Number or Quantity whatever, the Logarithm of x multiplied into y, gives the Logarithm of that Number; thus, if

$x = 4$, and $y = 3$, then will $x^y = 4^3 = 64$; and $y\,l\,x = l$: $64 = 1,8061800$. Here the Letter l denotes the Logarithm of the Quantity or Number to which it is prefixed, and the Equation is thus read; the Logarithm of x (4) multiplied into y (3) is equal to the Logarithm of 64, which is 1,8061800; for the Logarithm of 4 (x) is ,6020600, which being multiplied by 3 (y) the Product is 1,8061800, the Logarithm of 64, that is, $y \times$ Log. $x = 3 \times ,6020600$ ($=$ Log. of x^y) $=$ Log. of $64 = 1,8061800$, and, consequently, if 1,8061800 the Logarithm of 64, be divided by 3 (y) the Quotient will be ,6020600 the Lorithm of 4, equal to the Root x.

Though what is here delivered be only raising Powers and extracting Roots of given Numbers by Logarithms, yet it may suffice to give a Learner some Notion of Exponential Quantities; and how to solve various Equations containing such Quantities shall be shown in the Course of this Work.

SECTION II.

OF ADDITION.

IN the Addition of Algebraic Quantities there are three Cases, as follow:

CASE I.

To add Quantities that are alike, and have like Signs.

RULE.

Add the Co-efficients together, to their Sum join the common Letter or Letters, and prefix their proper Signs where necessary.

Ex. I.

	Ex. I.	Ex. II.	Ex. III.	Ex. IV.
Add	a	$-7b$	$4xy^2$	$2\sqrt{ax}$
	$2a$	$-b$	$7xy^2$	$5\sqrt{ax}$
	$3a$	$-2b$	$5xy^2$	$\sqrt{ax}$
Sum	$6a$	$-10b.$	$16xy^2.$	$8\sqrt{ax}.$

Obſerve, once for all, in Ex. I. I add the Co-efficients together, ſaying 3 and 2 is 5, and 1 makes 6, ſo that the Sum is $6a$, viz. 6 times a. And you muſt always remember, both in Addition and Subtraction, to count Unity, or 1, for the Co-efficient of every Term that has not a Figure prefixed to it.

CASE II.

To add Quantities that are like, but have unlike Signs.

RULE.

Subtract the leſs Co-efficient from the greater, to the Remainder prefix the Sign of the greater Co-efficient, and join their common Letters or Quantities. †

	Ex. I.	Ex. II.	Ex. III.	Ex. IV.	Ex. V.
To	$+6a$	$-7b$	$+2c$	$-cd$	$+3\sqrt{a^2+b^2}$
Add	$-2a$	$+6b$	$-2c$	$+3cd$	$-8\sqrt{a^2+b^2}$
Sum	$+4a.$	$-b.$	*	$+2cd.$	$-5\sqrt{a^2+b^2}$

In Ex. III. the Co-efficients of the two Quantities, viz. $+2c$ and $-2c$, are equal to each other, therefore they deſtroy one another, and ſo their Sum make 0 or *; which

† Here we have ſubmitted to Cuſtom, but (as Mr. Martin obſerves) it is with ſome Impropriety that we talk of adding Quantities with unlike Signs, ſince the Operation does, chiefly, conſiſt in Subtraction, as it muſt from the Nature of the Signs.

which Character in Algebra is frequently used to signify a vacant Place.

	Ex. VI.	Ex. VII.	Ex. VIII.	Ex. IX.	Ex. X.
To	$+2a$	$-6b$	$+7c$	$-3cd$	$+8\sqrt{a^2+b^2}$
Add	$-6a$	$+7b$	$-4c$	$+\ cd$	$-3\sqrt{a^2+b^2}$
Sum	$-4a.$	$+\ b.$	$+3c.$	$-2cd.$	$+5\sqrt{a^2+b^2}$

When many Quantities of the same Denomination are to be added together, whereof some are affirmative and others negative, reduce them first to two Terms, by adding all the affirmative Quantities together, and all the negative ones, and then add the two Terms into one, by Case II.

Thus, add $4a^2+7a^2-3a^2+12a^2-8a^2+a^2-5a^2$, together.

First, $4a^2+7a^2+12a^2+a^2=24a^2$, the Sum of the affirmative Quantities, and $-3a^2-8a^2-5a^2=-16a^2$, Sum of these negative; then $24a^2-16a^2=8a^2$, the Sum of the Whole.

EXAMPLE VI.

Add $5ax^2-4ax^2+10ax^2-8ax^2-6ax^2$, together:

First, $5ax^2+10ax^2=15ax^2$, and $-4ax^2-8ax^2-6ax^2=-18ax^2$; therefore the Sum of all these Quantities is $+15ax^2-18ax^2=-3ax^2$.

CASE III.

To add Quantities that are unlike.

RULE.

Set them all down one after another, with their Signs and Co-efficients prefixed.

EXAMPLE.

EXAMPLE.

To	$3a$	$-8xy$	$+2\sqrt{a^2-b^2}$
Add	$2b$	$+5y$	-9

Sum $3a+2b-8xy+5y+2\sqrt{a^2-b^2}-9.$

Here follows one Example wherein all the three Cafes of Addition are exercifed.

$3a^2$	$-9b^3$	$+9b^3$	$+xy$
$4a^2$	$+7\sqrt{ab}$	$-5d$	$-2y^3$
a^2	$-12b^3$	$-4\sqrt{ab}$	$+10$
$7a^2$	$+5b^3$	$+6c^2$	-6

Sum $15a^2 - 7b^3 + 3\sqrt{ab} + 6c^2 - 5d + xy - 2y^3 + 4.$

Here the firft Row is compofed of like Quantities, which are added together by Cafe I. The Terms $-9b^3$ and $+9b^3$ deftroy one another; and the Sum of $-12b^3$ and $+5b^3$ is $-7b^3$ (negative) by Cafe II. becaufe $-12b^3$ the Term with the greater Co-efficient is negative. The Sum of $+7\sqrt{ab}$ and $-4\sqrt{ab}$ is $+3\sqrt{ab}$ (affirmative) becaufe $+7\sqrt{ab}$ the Quantity with the greater Co-efficient is affirmative.

In like Manner, $+10$ and -6 together make $+4$; and the Reft of the Terms being unlike, they are therefore fet down with their refpective Signs and Co-efficients, conformable to Cafe III.

SECTION III.

OF SUBTRACTION.

RULE.

Change the Sign of the Quantity to be ſubtracted, then add them both together (by the preceding Rules of Addition) and the Sum will be the true Remainder.

	Ex. I.	Ex. II.	Ex. III.	Ex. IV.
From	$5a$	$3a$	$+4b$	$-c$
Take	$2a$	$5a$	$-2b$	$+8c$
Remains	$3a.$	$-2a.$	$+6b.$	$-9c.$

	Ex. V.	Ex. VI.	Ex. VII.	Ex. VIII.
From	$+8a^2$	$-8bc$	$+8c^2$	$-4d^3$
Take	$-8a^2$	$+8bc$	$+8c^2$	$+2d^3$
Remains	$+16a^2.$	$-16bc.$	*	$-6d^3.$

	Ex. IX.	Ex. X.	Ex. XI.	Ex. XII.
From	$-7x^2$	$-12xy$	$+14a^2b^2$	$5ax^2$
Take	$-5x^2$	$-12xy$	$-43a^2b^2$	$2ax^2+4$
Remainder	$-2x^2$	*	$+57a^2b^2$	$3ax^2-4$

The Eleven foregoing Examples of ſimple Quantities being obvious, I therefore paſs by them; but ſhall illuſtrate the Twelfth Example, in order to the ready underſtanding of thoſe which follow: In the Twelfth Example the compound Quantity $2ax^2+4$, being taken from the ſimple

simple Quantity $5ax^2$, the Remainder is $3ax^2-4$, and it is plain that the more there is taken from any Number or Quantity, the less will be left, and the less there is taken the more will be left: Now if only $2ax^2$, were taken from $5ax^2$, the Remainder would be $3ax^2$, and consequently if $2ax^2+4$, (which is greater than $2ax^2$ by 4) be taken from $5ax^2$, the Remainder will be less than $3ax^2$, by 4, that is, there will remain $3ax^2-4$, as above. For by changing the Sign of the Quantity $2ax^2+4$, and adding it to $5ax^2$, the Sum is $5ax^2-2ax^2-4$, but here the Term $-2ax^2$ destroys so much of $5ax^2$ as is equal to itself, and so $5ax^2-2ax^2-4$ becomes equal to $3ax^2-4$, by the general Rule for Subtraction.

	Ex. XIII.	Ex. XIV.	Ex. XV.
From	$5\,ax^2-$	$a+b$	$5a^2+3bx+14$
Take	$2\,ax^2-4$	$a-b+c$	$a^2+2bx-10$
Remainder	$3\,ax^2+4$	$*+2b-c.$	$4a^2+bx+24.$

	Ex. XVI.	Ex. XVII.	Ex. XVIII.
From	$9\sqrt{ax}-5a.$	$6\sqrt{a^2+b^2}$	$2\sqrt{x}+x^2$
Take	$6\sqrt{ax}$	$9\sqrt{a^2+b^2}-5a$	$\sqrt{x}+y^2$
Remain.	$3\sqrt{ax}-5a.$	$-3\sqrt{a^2+b^2}+5a.$	$\sqrt{x}+x^2-y^2.$

SECTION IV.

OF MULTIPLICATION.

IN Multiplication of Algebraic Quantities there is one general Rule for the Signs, namely, when the Signs of the Factors are both affirmative, or both negative, the Product will be affirmative; but if one of the Factors be affirmative

affirmative and the other negative, then the Product will be negative.

If the Quantities have numeral Co-efficients, multiply them together, and join their Product to the Product of the Letters or Quantities.

CASE I.

Of multiplying Quantities with like Signs.

EXAMPLES.

Multiply	a	$-3b$	$4ab$	$-5cd$
by	b	$-2c$	3	$-4x$
Product	$ab.$	$+6bc.$	$12ab.$	$+20cdx.$

CASE II.

Of multiplying Quantities with unlike Signs.

EXAMPLES.

Multiply	$-a$	$3b$	$-4ab$	$5cd$
by	b	$-2c$	3	$-4x$
Product	$-ab.$	$-6bc.$	$-12ab.$	$-20cdx.$

When several Quantities are to be multiplied together, they may be placed in any Order, and the value of the Product will be still the same; thus $cn = nc$: for cn and nc differ no more from one another than 3 times 5 do from 5 times 3.

Again, $abc = bac = cab = acb = bca = cba$; for all these are equal; and the same is to be observed of others: However, we frequently give those Letters the Precedency in a Product, that have it in the Alphabet.

CASE

CASE III.

To multiply any Power by another of the same Root.

RULE.

Add the Exponent of the Multiplier to that of the Multiplicand, and the Sum will be the Exponent of their Product (this in effect is merely Notation, and has been premised): Thus the Product of a^5 multiplied into a^3, is a^{5+3}, or a^8; that of x^n into x, is x^{n+1}; that of x^n into x^2, is x^{n+2}; that of x^m into x^n, is x^{m+n}; and that of cy^{n+2r} into y^{n-r}, is $cy^{n+2r+n-r}$, or cy^{2n+r}.

N. B. The Quantity c is a Co-efficient.

Again, the Product of $\overline{a+x}|^r$ multiplied into $a+x$, is $\overline{a+x}|^{r+1}$, and that of $\overline{x+y}|^n$ into $\overline{x+y}|^r$, is $\overline{x+y}|^{n+r}$.

This Rule is equally applicable to the Multiplication of Surds; for Surd Roots are fractional Powers of Quantities, which may be denoted by fractional Exponents (as has been defined in Notation), and the Sum of the fractional Exponents of any Number of Roots (of the same Quantity) will be the Exponent of their Product.

Thus the Product of $a^{\frac{1}{2}}$ multiplied into $a^{\frac{1}{2}}$, is $a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^{\frac{2}{2}} = a^1 = a$: In like Manner, $x^{\frac{1}{3}} \times x^{\frac{1}{3}} \times x^{\frac{1}{3}} = x^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = x^{\frac{3}{3}} = x^1 = x$, &c.

In the first of these Examples, the Sum of the Exponents is 1, for two Halves are equal to 1; and the Sum of the Exponents in the second Example is three Thirds, which are likewise equal to 1; and the Products are (a and x) both rational Quantities. Hence you may observe, that, if a surd Square Root be multiplied into itself, the Product will be rational; and a surd Cube Root multiplied into itself, and that Product multiplied again into the

the same Root, gives a rational Product. And in general, when the Sum of the Numerators of the Exponents is divisible by the common Denominator, without a Remainder, the Product will be rational.

Thus, $a^{\frac{5}{4}} + a^{\frac{3}{4}} = a^{\frac{5}{4}+\frac{3}{4}} = a^{\frac{5+3}{4}} = a^{\frac{8}{4}} = a^{2}$:

Here the Quantity $a^{\frac{8}{4}}$, is reduced to a^2, by actually dividing 8, the Numerator of the Exponent by its Denomitor 4; and the Sum of the Exponents considered merely as vulgar Fractions is $\frac{5}{4} + \frac{3}{4} = \frac{8}{4} = 2$. When the Sum of the Numerators and the Denominator of the Exponents admit of a common Divisor, greater than Unity, then the Exponent of the Product may always be reduced, like a vulgar Fraction, to lower Terms, retaining still the same Value.

Thus $x^{\frac{4}{9}} \times x^{\frac{2}{9}} = x^{\frac{6}{9}} = x^{\frac{2}{3}}$. But $x^{\frac{2}{5}} \times x^{\frac{2}{5}} = x^{\frac{4}{5}}$:

Here, it is obvious, that the Exponent of the Product cannot be reduced to lower Terms. Compound Surds (of the same Quantity) are multiplied in the very same Manner as simple ones: Thus $\overline{a+x}|^{\frac{1}{2}} \times \overline{a+x}|^{\frac{1}{2}} = \overline{a+x}|^{\frac{2}{2}} = \overline{a+x}|^{1} = a+x$; $\overline{a^2+x^2}|^{\frac{2}{3}} \times \overline{a^2+x^2}|^{\frac{1}{3}} = \overline{a^2+x^2}|^{\frac{3}{3}} = a^2+x^2$.

So likewise $\sqrt[8]{a+x} \times \sqrt[8]{a+x} = \sqrt[4]{a+x} = \overline{a+x}|^{\frac{1}{4}}$

$\sqrt[4]{a+x} \times \sqrt[4]{a+x} = \sqrt{a+x} = \overline{a+x}|^{\frac{1}{2}}$;

and $\sqrt{a+x} \times \sqrt{a+x} = a+x$.

These Examples are chiefly intended to define the Grounds on which the Products of Surds become rational.

CASE IV.

Different Quantities under the same radical Sign are multiplied together like rational Quantities, only the Product (if it does not become rational) must stand under the same radical Sign.

Thus

Thus, $\sqrt{7} \times \sqrt{3} = \sqrt{7 \times 3} = \sqrt{21}$;
$3\sqrt{a} \times 2\sqrt{b} = 6\sqrt{ab}$; $9\sqrt[3]{7cx} \times 5\sqrt[3]{2y} =$
$45\sqrt[3]{14cxy}$; and $5a\sqrt[n]{4d} \times 3b\sqrt[n]{2cd} =$
$15ab\sqrt[n]{8cd^2} = 15ab \times \overline{8cd^2}|^{\frac{1}{n}}$.

It may not be improper to obferve, that unequal Surds have fometimes a rational Product. As, $\sqrt{32} \times \sqrt{2} = \sqrt{64} = 8$; $\sqrt{12ab} \times \sqrt{3ab} = \sqrt{36a^2b^2} = 6ab$; $a\sqrt[3]{x^2y} \times b\sqrt[3]{xy^2} = ab\sqrt[3]{x^3y^3} = abxy$;
and $\sqrt[n]{\overline{a+x}|^{n-r}} \times \sqrt[n]{\overline{a+x}|^{r}} = \sqrt[n]{\overline{a+x}|^{n-r+r}}$.
$= \sqrt[n]{\overline{a+x}|^{n}} = a + x$

CASE V.

To multiply a compound Quantity by a fimple one.

RULE.

Multiply every Term of the Multiplicand by the Multiplier.

EXAMPLES.

Multiply	$5a + bc$	$3\sqrt{ab} - 4b^2 + 7c\sqrt{ax}$
by	$3c$	$2b$
Product	$15ac + 3bc^2$.	$6b\sqrt{ab} - 8b^3 + 14bc\sqrt{ax}$.

Multiply $3x - 4\sqrt{n} + 5d\sqrt{x^2 - y^2} - 6b\sqrt{c}$
by $2a\sqrt{c}$.

Product $6ax\sqrt{c} - 8a\sqrt{cn} + 10ad\sqrt{cx^2 - cy^2} - 12abc$.

CASE

CASE VI.

To multiply compound Quantities together.

RULE.

Multiply every Part of the Multiplicand by each Term in the Multiplier, then add all the Products into one Sum; and that Sum will be the Product required.

EXAMPLES.

Multiply	$a + b$	$a + b$
by	$a + b$	$a - b$
	$a^2 + ab$ $\quad + ab + b^2$	$a^2 + ab$ $\quad - ab - b^2$
Product	$a^2 + 2ab + b^2.$	$a^2 \quad * - b^2$

In the first Example, I multiply $(a + b)$ the Multiplicand into a the first Term of the Multiplier, and the Product is $a^2 + ab$; then I multiply the Multiplicand into b the second Term of the Multiplier, and the Product is $ab + b^2$: the Sum of these two Products is $a^2 + 2ab + b^2$, as above, and is the Square of $a + b$. In the first Example, the like Terms of the Product, viz. ab and ab together make $2ab$; but in the second Example the Terms $+ ab$ and $- ab$ (having contrary Signs) destroy each other, and the Product is $a^2 - b^2$ the Difference of the Squares of a and b. Hence it appears, that the Sum and Difference of two Quantities multiplied together, produce the Difference of their Squares. And, by the next following Example you may observe, that the Square of the Difference of two Quantities (as a and b) is equal to $(a^2 - 2ab + b^2)$ the Sum of their Squares, minus (or less $2ab$) twice their Product. See the Operation.

Multiply

Multiply	$a - b$	$a - b$
by	$a - b$	$c - d$
	$a^2 - ab$	$ac - bc$
	$\quad - ab + b^2$	$\quad - ad + bd$
Product	$a^2 - 2ab + b^2$	$ac - bc - ad + bd.$
Multiply	$\sqrt{a} + \sqrt{b}$	$\sqrt{a} + \sqrt{b}$
by	$\sqrt{a} + \sqrt{b}$	$\sqrt{a} - \sqrt{b}$
	$a + \sqrt{ab}$	$a + \sqrt{ab}$
	$\quad + \sqrt{ab} + b$	$\quad - \sqrt{ab} - b$
Product	$a + 2\sqrt{ab} + b.$	$a \quad * - b.$
Multiply	$7x - 4$	$\sqrt[4]{bc} + 6$
by	$2y - 3$	$\sqrt[4]{bc} - 6$
	$14xy - 8y$	$\sqrt{bc} + 6\sqrt[4]{bc}$
	$\quad - 21x + 12$	$\quad - 6\sqrt[4]{bc} - 36$
Product	$14xy - 21x - 8y + 12.$	$\sqrt{bc} \quad * \quad - 36.$

Multiply	$x^2 + 10xy + 7$
by	$x^2 - 6xy + 4$
	$x^4 + 10x^3y + 7x^2$
	$\quad - 6x^3y - 60x^2y^2 - 42xy$
	$\quad\quad + 4x^2 + 40xy + 28.$
Product	$x^4 + 4x^3y - 60x^2y^2 + 11x^2 - 2xy + 28.$

To prove that like Signs produce an affirmative, and unlike Signs a negative Product.

First, it is evident that $+x - x = 0$; therefore if $+x - x$, be multiplied by (any Number) n, the Product must

must be (o) nothing, because the Factor $+x-x$ is o. Now the first Term of the Product is $+nx$ (by Case I.) therefore the second Term of the Product must be $-nx$, which destroys $+nx$; so that the whole Product is $+nx-nx=0$ (nothing): and consequently $-x$ multiplied by $+n$, gives $-nx$. Again, if $+x-x=0$, be multiplied by $-n$, the first Term of the Product being $-nx$, the second Term must be $+nx$, because the two Terms together must destroy each other; otherwise their Amount could not be o, as it evidently must, since the Factor $+x-x$ is (o) nothing, and therefore $-x$ multiplied by $-n$ gives $+nx$.

I shall subjoin another Instance, which, perhaps, may be more intelligible and satisfactory to some Learners than that aforegoing. Wherein let it be required to multiply $a-b$ by $c-d$; Suppose $a=8$, $b=3$, $c=7$, and $d=4$: Then will the Factor $a-b=8-3=5$, and the Factor $c-a=7-4=3$. Now if our Rule for the Signs be true, the Product of $a-b$ multiplied by $c-d$, (being equal to the Product of 5 multiplied by 3) will be 15, and so we find it; For $\overline{a-b}\times\overline{c-d}=a\times c-b\times c-a\times d+b\times d=8\times7-3\times7-8\times4+3\times4=56-21-32+12=68-53=15$.

SECTION V.

OF DIVISION.

IN Division of Algebraic Quantities the Rule for the Signs is the same as in Multiplication, viz. If the Signs of the Divisor and Dividend are alike (that is, both +, or both —,) then the Sign of the Quotient must be +; but if they are unlike, the Sign of the Quotient must be —. This is evidently deduced from the Rule in Multiplication, if it be considered, that the Quotient must be such a Quantity as when multiplied by the Divisor, shall give the Dividend. This is a general Rule for all Operations in Division, and is only the Reverse of Multiplication.

Thus

CASES I and II.

Thus, it is evident *

Divisor. Dividend. Quotient.

1. That $\begin{cases} +a)\ +ab\ (+b\text{; because } +b \text{ multiplied by } +a \text{ gives } +ab\text{;} \\ -2c)-6bc\ (+3b \text{ and } +3b \text{ multiplied by } -2c \text{ gives } -6bc. \end{cases}$

2. That $\begin{cases} +3a)-12ab\ (-4b\text{; for } -4b \text{ multiplied by } +3a \text{ gives } -12ab\text{;} \\ -4c)+20bc\ (-5b \text{ and } -5b \text{ multiplied by } -4c \text{ gives } +20bc. \end{cases}$

In these four Examples all the Variations that can possibly happen with respect to the Signs are exhibited at one View; and the Division may be read thus, viz. +* ab divided by $+a$, gives $+b$; the Quotient of $-6bc$ by $-2b$ is $+3b$, &c. the other two Quotients (viz. $-4b$ and $-5b$) are negative, because each of the Divisors has a contrary Sign to that of its Dividend; and it appears that each Quotient being multiplied by its respective Divisor gives its corresponding Dividend.

It may be proper to observe, that when any Quantity is to be divided by itself, the Quotient will be unity or 1, because any thing contains itself once; thus $x \div x$ gives 1; and $\sqrt{2ab}$ divided by $\sqrt{2ab}$ gives 1; and so of other equal Quantities.

When the Factors of the Divisor are not comprised in the Dividend, then the Quotient must be expressed by a Fraction (as has been shewn in Notation) Thus the Quotient of a divided by c, is $\frac{a}{c}$; that of bcx^2 by adr, is $\frac{bcx^2}{adr}$; but the Quotient of $abcdx$, divided by abx is $\frac{abcdx}{abx} = \frac{cd}{1} = cd$, a whole Quantity; here the Fraction $\frac{abcdx}{abx}$ is reduced to cd, by dividing its

Numerator

Numerator and Denominator by $a\,b\,x$, and the Quotient, viz. $c\,d$, being multiplied by $a\,b\,x$, the Divisor, gives $a\,b\,c\,d\,x$ the Dividend; which shews that dividing the Numerator and Denominator of a Fraction by the same Quantity does not alter its Value. Therefore when the Dividend does not exactly contain all the Parts of the Divisor, but has some Factors common with it, cast away all such Factors out of the Dividend and Divisor as are common to both, or else cancel the common Factors out off the Numerator and Denominator of the fractional Quotient, so shall the Quotient (by each Method) be in its lowest Terms, and shall be of the same Value as if all the Factors were actually retained in its Numerator and Denominator.

First, let it be required to divide $a\,b\,c\,d\,x$ by $b\,d\,n\,r$, then casting away the common Factors, viz. $b\,d$, out of the Dividend and Divisor, the Quotient will be truly expressed by $\dfrac{a\,c\,x}{n\,r}$, and is evidently in its lowest Terms.

Again, $a\,b\,n$ divided by $a\,b\,d$, gives $\dfrac{a\,b\,n}{a\,b\,d} = \dfrac{n}{d}$;

$4\,a\,c\,x$ divided by $8\,a\,x\,y$, gives $\dfrac{4\,a\,c\,x}{8\,a\,x\,y} = \dfrac{c}{2\,y}$; and

$3\,a\,x\,y$ divided by $3\,a\,b\,c\,x\,y$, gives $\dfrac{3\,a\,x\,y}{3\,a\,b\,c\,x\,y} = \dfrac{1}{b\,c}$.

By this last Example it appears, that when all the Factors of the Dividend are contained in the Divisor, then the Numerator of the Quotient will be unity, or 1.

CASE III.

To divide any Power by another of the same Root.

RULE.

Subtract the Exponent of the Divisor from that of the Dividend, and the Remainder will be the Exponent of the Quotient. Thus the Quotient of a^8 divided by a^3 is a^{8-3}, or

or a^5; that of x^n by x, is x^{n-1}; that of x^n by x^2, is x^{n-2}; that of x^{m+n} by x^n is x^m; and that of x^n by x^r is x^{n-r}.

But it is to be observed, that when the Exponent of the Divisor is greater than that of the Dividend, the Quotient will have a negative Exponent; thus the Quotient of x^5 divided by x^7, is x^{5-7}, or x^{-2}; and that of $a x^2$ by x^5, is $a x^{-3}$; and these Quotients (viz. x^{-2}, and $a x^{-3}$) are respectively equal to $\frac{1}{x^2}$ and $\frac{a}{x^3}$: For x^5 being actually divided by x^7, gives $\frac{x^5}{x^7} = \frac{1}{x^2}$; and $a x^2$ divided by x^5, gives $\frac{ax^2}{x^5} = \frac{a}{x^3}$ as above. In like Manner $a x^n$ divided by $c x^{2n}$ gives $\frac{ax^n}{cx^{2n}} = \frac{a}{cx^n}$, and the Quotient of $\overline{a^2+x^2}^{\,m}$ divided by $\overline{a^2+x^2}^{\,n}$, is $\overline{a^2+x^2}^{\,m-n}$. Moreover, $a^{\frac{3}{2}}$ divided by $a^{\frac{1}{2}}$, gives $a^{\frac{3-1}{2}} = a^{\frac{2}{2}} = a$; $\overline{a+x}^{\frac{5}{9}}$ divided by $\overline{a+x}^{\frac{2}{9}}$, gives $\overline{a+x}^{\frac{5-2}{9}} = \overline{a+x}^{\frac{3}{9}} = \overline{a+x}^{\frac{1}{3}}$; and $\overline{ab+x^2}^{\frac{m}{n}}$ divided by $\overline{ab+x^2}^{\frac{r}{n}}$, gives $\overline{ab+x^2}^{\frac{m-r}{n}}$.

SCHOLIUM.

When the Exponents have not the same Denominator, they may be brought to a common Denominator (like vulgar Fractions) and then their Numerators may be occasionally added, or subtracted, as before.

Thus, the Quotient of $\overline{ac+x}^{\frac{1}{2}}$ divided by $\overline{ac+x}^{\frac{1}{4}}$ is $\overline{ac+x}^{\frac{2}{4}-\frac{1}{4}} = \overline{ac+x}^{\frac{2-1}{4}} = \overline{ac+x}^{\frac{1}{4}}$.

Here the Exponent of the Dividend is brought to a common Denominator with that of the Divisor, by multiplying its Numerator and Denominator by 2.

CASE IV.

Surd Quantities under the same radical Sign, are divided by one another like rational Quantities, only the Quotient, if it does not become rational, must stand under the same radical Sign.

Thus the Quotient of $\sqrt{21}$ divided by $\sqrt{3}$, is $\sqrt{7}$; that of $\sqrt{ab}$ by $\sqrt{a}$, is $\sqrt{b}$; that of $a^2\sqrt[3]{16c}$ by $a\sqrt[3]{2c}$, is $a\sqrt[3]{8}$, or $2a$; that of $6axy\sqrt[n]{(ax)^r}$ by $2\sqrt[n]{(ax)^r}$, is $3axy$; and that of $(12a^2x^3y^5)^{\frac{m}{n}}$, by $(3a^2x^2y^3)^{\frac{m}{n}}$, is $4xy^2\,^{\frac{m}{n}}$.

CASE V.

A compound Quantity is divided by a simple one, by dividing every Term thereof by the given Divisor.

Thus, $3c\,)\,15ac+3bc\,(\,5a+b$ Quotient;
Also, $4ab\,)\,8ab\sqrt{x}-12a^3b^2+4ab\,(\,2\sqrt{x}-3a^2b+1$ Quotient.

CASE VI.

But if the Divisor and Dividend be both compound Quantities, range the Terms according to the Dimensions of some Letter in them (See the following Example, where a is the first Term of the Divisor; and the third, second, and first Powers of the Letter a, are successively ranged one after another in the Dividend; but in the last Term of the Divisor and Dividend, a is not included) then divide the first Term of the Dividend by the first Term of the Divisor; place the Result in the Quotient, and multiply the whole Divisor thereby; subtract the Product from its respective Terms of the Dividend; to the Remainder bring down the next Term, or Terms, of the Dividend; call the Sum a Dividual, and divide the first Term of the Dividual by the first Term of the Divisor, put

put the Result likewise in the Quotient, then multiply and subtract as before, and repeat the Process till all the Terms of the Dividend are brought down and divided. If there be a Remainder, you are to proceed after the same Manner till no Remainder is left; or till it appears that there will be always some Remainder.

EXAMPLE.

Let it be required to divide $a^3 - 3a^2x - 3ax^2 + x^3$ by $a + x$.

See the Operation.

$$\begin{array}{rll}
a+x) & a^3 - 3a^2x - 3ax^2 + x^3 & (a^2 - 4ax + x^2 \\
 & a^3 + a^2x & \\
\hline
 & -4a^2x - 3ax^2 & \text{first Dividual;} \\
 & -4a^2x - 4ax^2 & \\
\hline
 & +ax^2 + x^3 & \text{second Dividual.} \\
 & +ax^2 + x^3 & \\
\hline
 & 0 \quad 0 & \\
\hline
\end{array}$$

An Explication of the preceding Work.

First, a^3 divided by a, gives a^2 for the first Term of the Quotient, by which I multiply the whole Divisor, viz. $a+x$, and the Product is $a^3 + a^2x$, which, being taken from the two first Terms of the Dividend, leaves $-4a^2x$; to this Remainder I bring down $-3ax^2$, the next Term of the Dividend, and the Sum is $-4a^2x - 3ax^2$ the first Dividual; now dividing $-4a^2x$ the first Term of this Dividual by (a) the first Term of the Divisor, there comes out $-4ax$ (a negative Quantity) which I also put in the Quotient, and multiplying the whole Divisor by it, the Product is $-4a^2x - 4ax^2$, which being taken from the first Dividual, the Remainder is $+ax^2$, to which I bring down x^3, the last Term of the Dividend, and the Sum is $+ax^2 + x^3$, the second Dividual, and $+ax^2$ (the

first Term of the second Dividual) divided by (a) the first Term of the Divisor, gives x^2 for the last Term of the Quotient, by which I multiply the whole Divisor, and the Product is $+ax^2+x^3$, which being taken from the second Dividual leaves nothing; and the Quotient required is $a^2-4ax+x^2$.

Other Examples in Division may be as follow:

Divide $a^5+a^4x-a^3x^2-7a^2x^3-6x^5$ by a^2-x^2.

See the Work.

$$
\begin{array}{lllll}
a^2-x^2) & a^5+a^4x-a^3x^2-7a^2x^3-6x^5 & (a^3+a^2x-6x^3. \\
 & a^5 \quad -a^3x^2 & \\
\hline
 & +a^4x \quad -7a^2x^3 & \\
 & +a^4x \quad -a^2x^3 & \\
\hline
 & \quad -6a^2x^3-6x^5 & \\
 & \quad -6a^2x^3-6x^5 & \\
\hline
 & \quad\quad 0 \quad 0 &
\end{array}
$$

Divide $a-b-c\sqrt{a}+c\sqrt{b}$ by $\sqrt{a}-\sqrt{b}$.

$$
\begin{array}{lll}
\sqrt{a}-\sqrt{b}) & a-b-c\sqrt{a}+c\sqrt{b} & (\sqrt{a}+\sqrt{b}-c. \\
 & a-\sqrt{ab} & \\
\hline
 & +\sqrt{ab}-b & \\
 & +\sqrt{ab}-b & \\
\hline
 & \quad -c\sqrt{a}+c\sqrt{b} & \\
 & \quad -c\sqrt{a}+c\sqrt{b} & \\
\hline
 & \quad\quad 0 \quad 0 &
\end{array}
$$

It may not be unnecessary to elucidate this Operation, as it consists of both rational and surd Quantities.

Here (a) the first Term of the Dividend, being divided by $\sqrt{a}$, the first Term of the Divisor, gives $\sqrt{a}$ for the first

first Term of the Quotient. For $a = a^1$, and $\sqrt{a} = a^{\frac{1}{2}}$ and the Difference of these two Exponents is $(1 - \frac{1}{2}$ or$)$ $\frac{1}{2}$; therefore a^1 divided by $a^{\frac{1}{2}}$, gives $a^{1-\frac{1}{2}} = a^{\frac{1}{2}} = \sqrt{a}$, as above. Or it may be considered thus: Ask what Quantity being multiplied by $\sqrt{a}$ will give a, and the Answer is $\sqrt{a}$; then multiplying the Divisor by $\sqrt{a}$, the Product is $a - \sqrt{ab}$; but there being no Term in the Dividend that corresponds to $-\sqrt{ab}$ the second Term of this Product, therefore I subtract $a - \sqrt{ab}$ from $a - b$ the two first Terms of the Dividend, and changing the Sign of the Quantity $-\sqrt{ab}$ the Remainder is $+\sqrt{ab} - b$; (See the 12th Example in Subtraction) Now $+\sqrt{ab}$ the first Term of this Remainder divided by $\sqrt{a}$, the first Term of the Divisor, gives $-\sqrt{b}$ for the second Term of the Quotient, by which I multiply the Divisor, and subtracting the Product, viz. $+\sqrt{ab} - b$ from the aforesaid Remainder, nothing remains; then I bring down $-c\sqrt{a} + c\sqrt{b}$, the two last Terms of the Dividend, and $-c\sqrt{a}$ the first of these Terms being divided by $\sqrt{a}$ (the first Term of the Divisor) gives $-c$ for the last Term of the Quotient; then multiplying the Divisor by $-c$, and subtracting the Product from the two last Terms of the Dividend, nothing remains, and the whole Quotient is $\sqrt{a} + \sqrt{b} - c$.

Again, divide $2a^4 - 32$ by $a - 2$.

$$
\begin{array}{rl}
a-2)\ 2a^4-32 & (2a^3+4a^2+8a+16. \\
2a^4-4a^3 & \\
\hline
+4a^3-32 & \\
+4a^3-8a^2 & \\
\hline
+8a^2-32 & \\
+8a^2-16a & \\
\hline
+16a-32 & \\
+16a-32 & \\
\hline
0\quad 0 & \\
\hline
\end{array}
$$

Divide a^3+x^3 by $a+x$.

$$
\begin{array}{r|l}
a+x) & a^3+x^3\ (a^2-ax+x^2, \\
 & a^3+ax \\ \hline
 & -ax+x^3 \\
 & -ax-ax^2 \\ \hline
 & +ax^2+x^3 \\
 & +ax^2+x^3 \\ \hline
 & 0 \quad 0
\end{array}
$$

Divide $x^3+3x^2y+3xy^2+y^3$ by $x^2+2xy+y^2$.

$$
\begin{array}{r|l}
x^2+2xy+y^2) & x^3+3x^2y+3xy^2+y^3\ (x+y. \\
 & x^3+2x^2y+xy^2 \\ \hline
 & x^2y+2xy^2+y^3 \\
 & x^2y+2xy^2+y^3 \\ \hline
 & 0 \quad 0 \quad 0
\end{array}
$$

Divide $a^3-3a^2c+4ac^2-2c^3$ by $a^2-2ac+c^2$.

$$
\begin{array}{r|l}
a^2-2ac+c^2) & a^3-3a^2c+4ac^2-2c^3\ \left(a-c+\dfrac{ac^2-c^3}{a^2-2ac+c^2}\right. \\
 & a^3-2a^2c+\ ac^2 \\ \hline
 & -\ a^2c+3ac^2-2c^3 \\
 & -\ a^2c+2ac^2-\ c^3 \\ \hline
\text{Remains} & +\ ac^2-\ c^3
\end{array}
$$

Here it is obvious that the Divifion cannot terminate without a Remainder; therefore I write the Divifor under the Remainder, with a Line between them, and add the Fraction to $a-c$ the other two Terms, which together make $\left(a-c+\dfrac{ac^2-c^3}{a^2-2ac+c^2}\right)$ the whole Quotient.

But

But when the Dividend does not precisely contain the Divisor, then we generally express the whole Quotient fractionwise, casting away all such Letters or Factors (if any such there be) as are found in every Term of the Numerator and Denominator of the Quotient.

Thus $a^2bx+acx^2+ax^3$ divided by $adx+anx$,

Gives $\dfrac{a^2bx+acx^2+ax^3}{adx+anx}$, or $\dfrac{ab+cx+x^2}{d+n}$.

Here the Quotient $\dfrac{a^2bx+acx^2+ax^3}{adx+anx}$, is reduced to $\dfrac{ab+cx+x^2}{d+n}$, by dividing every Term of its Numerator and Denominator by ax.

Lastly, $a+ab+d^2$ divided by $a^2-ac+a^2c^2$, gives $\dfrac{a+ab+d^2}{a^2-ac+a^2c^2}$.

Here the Quotient cannot be reduced to lower Terms, because the Factor a is not to be found in the Term d^2.

But it is to be observed, that though a Fraction cannot be reduced to lower Terms by a simple Divisor, yet it may sometimes by a compound one; as shall be shewn in the next Section.

SECTION VI.

CASE I.

To reduce an Algebraic Fraction to its lowest Terms.

RULE.

Divide its Numerator and Denominator by their greatest common Divisor.

Thus, dividing both the Numerator and Denominator of the Fraction $\frac{2ab}{4bc}$ by $2b$, it is reduced to $\frac{a}{2c}$ its lowest Terms; and $\frac{a^2b^2c}{a^3b}$ divided in like Manner by $a^2 b$, is reduced to $\frac{bc}{a}$; moreover, $\frac{6a^2b-12ax}{6a^2x}$ divided by $6a$, is reduced to $\frac{ab-2x}{ax}$, and dividing every Term of the Numerator and Denominator of the Fraction $\frac{12a^3x^3-8a^2x^2+16ax^2}{4acx^2+20a^2x^3y}$ by $4ax^2$, it will be reduced to $\frac{3a^2x-2a+4}{c+5axy}$.

The simple Divisors by which these Fractions are reduced, were had by Inspection, and compound Divisors, by which Fractions can sometimes be reduced to their least Terms, may be found as follow:

First, divide that Quantity consisting of the highest Powers by the other, and the last Divisor by the last Remainder, and so on continually till nothing remains, and the last Divisor will be the greatest common Measure, but observe to divide the Remainders that arise in the Operation by their greatest simple Divisors, or others that are prime to the Divisors from which the Remainders arose; and always range the Terms of both Quantities according to the Dimensions of the Letter that has the highest Powers.

For Example, reduce the Fraction $\frac{x^5+5ax^4-a^2x^2-5a^3x}{x^4+3ax^3-a^2x-3a^3}$ to its lowest Terms. First dividing the Quantity composed of the highest Powers of x by the other, the Work will stand thus:

$$x^4+3ax^3-a^2x-3a^3)\ x^5+5ax^4-a^2x^2-5a^3x\ (x$$
$$\underline{x^5+3ax^4-a^2x^2-3a^3x}$$

and the Remainder is $2ax^4 \quad -2a^3x$, which being divided by $2ax$, its greatest simple Divisor, gives x^3-a^2, then

$$\begin{array}{l} x^3-a^2)\ x^4+3ax^3-a^2x-3a^3\ (x+3a, \text{ the new Denominator.} \\ \quad\ \ x^4 \qquad\quad -a^2x \\ \hline \qquad +3ax^3 \qquad -3a^3 \\ \qquad +3ax^3 \qquad -3a^3 \\ \hline \qquad\quad 0 \qquad\quad 0 \end{array}$$

$$\begin{array}{l} \text{and } x^3-a^2)\ x^5+5ax^4-a^2x^2-5a^3x\ (x^2+5ax, \text{ the new Numerator.} \\ \qquad\qquad x^5 \qquad\quad -a^2x^2 \\ \hline \qquad\qquad +5ax^4-5a^3x \\ \qquad\qquad +5ax^4-5a^3x \\ \hline \qquad\qquad\quad 0 \qquad 0 \\ \hline \end{array}$$

Therefore x^3-a^2 is the greatest common Measure, and hence the Fraction proposed is reduced to $\frac{x^2+5ax}{x+3a}$ its lowest Terms.

In this Example it appears that the Remainder, namely $2ax^4-2a^3x$, is too great by $2ax$ Times to measure $x^4+3ax^3-a^2x-3a^3$, the Divisor and Dividend $x^5+5ax^4-a^2x^2-5a^3x$; therefore if each of these Terms be multiplied by $2ax$, the Remainder (without being reduced) will divide them, and the Quotients will come out exactly the same; and if it happens that the first Term of the Divisor does not exactly measure that of the Dividend, the whole Dividend or Divisor, or both, may be first of all either multiplied or divided by any Quantity that will make the Operation succeed. The Ground of this Reduction is, that any Quantity which measures the Divisor and Dividend, must also measure the Remainder (if there is any) for the Divisor being multiplied into the Quotient, the Product will precisely contain the common Measure a certain Number of Times, and is therefore a Multiple of it, which being taken from the Dividend (if it is not equal to it), must evidently leave a Remainder, in which the common Divisor will be always included. But if by proceeding as above directed, there happens to be a Remainder of one Term

Term only, then we may conclude, that the Fraction proposed is already in its lowest Terms, or that it does not admit of a compound Divisor, which will divide both its Numerator and Denominator without a Remainder.

The preceding Directions extend to Fractions composed of two different Letters only; but sometimes Divisors can be found, by which Fractions consisting of more Letters may be reduced. This is done by putting the Numerator or Denominator (in which ever a common Measure can be best discovered) into two Parts, and finding a Divisor which will measure each of those Parts; for that Divisor will evidently measure the whole of that Quantity; by the same Divisor the other Quantity is to be tried, and if it will divide it, then the Fraction may be reduced to lower Terms.

For instance, let it be required to reduce the Fraction $\frac{ax^4y+2x^3y^2-4a^3x^2z-8a^2xyz}{ax^2+3a^2xy^2+2xy+6ay^3}$ to its least Terms.

First, I divide the Denominator into two Parts, viz. ax^2+2xy and $3a^2xy^2+6ay^3$; here it is obvious that $\overline{ax+2y}$ is a Divisor to both the Parts; for $ax^2+2xy=\overline{ax+2y}\times x$, and $3a^2xy^2+6ay^3=\overline{ax+2y}\times 3ay^2$; the Sum of these two Parts may be expressed by $\overline{ax+2y}\times\overline{x+3ay^2}$; hence it appears that the Factor $x+3ay^2$, will likewise measure the Denominator: Now if the given Fraction can be reduced to lower Terms, one of these two Factors must divide its Numerator; and by trying with the former (namely $ax+2y$) I find it succeed, and the Quotient is x^3y-4a^2xz, whence the Fraction proposed is reduced to $\frac{x^3y-4a^2xz}{x+3ay^2}$, and is in its lowest Terms. In this Example the common Measure might have been found by dividing the two Parts above specified by (x, and $3ay^2$) their respective simple Divisors, and also by dividing the first two Terms of the Numerator by x^3y, their greatest simple Divisor.

CASE

CASE II.

To reduce Fractions of different Denominators to Fractions of equal Value that shall have a common Denominator.

RULE.

Multiply each Numerator, separately taken, into all the Denominators but its own, and the Products shall give the new Numerators. Then multiply all the Denominators into one another, and the Product shall be the common Denominator.

Thus let it be required to reduce $\frac{b}{c}$ and $\frac{x}{a}$ to the same Denominator, then $a \times c = ac$, is the common Denominator, and $a \times b$, $c \times x = ab$, cx, are the new Numerators, under each of which write (ac) the common Denominator, and you will have $\frac{ab}{ac}$ and $\frac{cx}{ac}$ for the Fractions required.

In the same Manner the Fractions $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{n}{s}$ will be reduced to $\frac{ads}{bds}$, $\frac{bcs}{bds}$, and $\frac{bdn}{bds}$.

If the given Denominators admit of a common Divisor, divide them by it, then multiply each Numerator by all the other Fractions' Denominators so divided (except its own) for new Numerators, and multiply all the Quotients into the Divisor for a common Denominator.

Thus, reduce $\frac{b}{ac}$, and $\frac{n}{ad}$, to the same Denominator; here dividing the given Denominators by a, we have bd and cn for the new Numerators, and $a \times c \times d$, or acd for the common Denominator, hence the Fractions are reduced to $\frac{bd}{acd}$, and $\frac{cn}{acd}$.

In

In like Manner the Fractions $\frac{c}{2bx}$ and $\frac{a}{4b+2b^3d}$, will be reduced to $\frac{2c+cb^2d}{4bx+2b^3d}$ and $\frac{ax}{4bx+2b^3d}$. Here the greatest common Divisor of the given Denominators is $2b$.

If the given Denominators had not been previously divided, the Fractions would have been of the same Value, but in higher Terms; thus in the first of these Examples, the Fractions after Reduction without dividing their Denominators, would have been $\frac{abd}{a^2cd}$ and $\frac{acn}{a^2cd}$, which are respectively equal to $\frac{bd}{acd}$ and $\frac{cn}{acd}$, but as it is always best to reduce the Fractions at once to the least Terms that the Case will admit of, therefore divide the given Denominators at first where it is convenient, or conceive them to be divided, by neglecting the common Factors in your Mind, and work with the Rest as before.

CASE III.

To reduce an Integer to an equivalent Fraction of a given Denominator.

RULE.

Multiply the Integer by the given Denominator, and under the Product write the same Denominator.

Thus, let a have the Denominator x, then $\frac{a \times x}{x}$, or $\frac{ax}{x}$ is the Fraction required: Again, let $a+b$ have the Denominator d, then $\frac{ad+bd}{d}$ is the Fraction sought.

CASE

CASE IV.

To reduce a mixed Quantity to an improper Fraction.

RULE.

Multiply the Part that is an Integer by the Denominator of the fractional Part, and to the Product add the Numerator, under their Sum write the former Denominator.

Thus $a + \frac{b}{x}$ reduced to an improper Fraction, gives $\frac{ax+b}{x}$; also $a-x+\frac{n}{d} = \frac{ad-dx+n}{d}$; $a-x+\frac{a^2-ax}{x}$ $= \frac{ax-x^2+a^2-ax}{x} = \frac{a^2-x^2}{x}$; and $a-x+\frac{4x^2}{a+x} =$ $\frac{a^2-x^2+4x^2}{a+x} = \frac{a^2+3x^2}{a+x}$.

CASE V.

To reduce an improper Fraction to a whole or mixed Number.

RULE.

Divide the Numerator by its Denominator, as far as you can, and the Quotient shall give the integral Part, then write the Denominator of the given Fraction under the Remainder for the fractional Part.

Thus $\frac{ax+b}{x} = \frac{ax}{x} + \frac{b}{x} = a + \frac{b}{x}$; again, reduce $\frac{a^2+3x^2}{a+x}$ to its proper Quantity.

$$\begin{array}{l} a+x)\ a^2+3x^2\ (a-x+\dfrac{4x^2}{a+x}. \\ \quad\quad a^2+ax \\ \hline \quad\quad -ax+3x^2 \\ \quad\quad -ax-x^2 \\ \hline \quad\quad\quad +4x^2 \\ \hline \end{array}$$

CASE VI.

To add and subtract Fractions.

RULE.

Reduce them to a common Denominator, and add or subtract the new Numerators; the Sum or Difference set over the common Denominator, is the Sum or Remainder required.

Thus, add $\frac{a}{c}$ to $\frac{b}{x}$, then $\frac{ax}{cx}+\frac{bc}{cx}=\frac{ax+bc}{cx}$, the Sum; add $\frac{b}{3a}$, $\frac{x}{5c}$ and $\frac{n}{d}$ together, then $\frac{5bcd}{15acd}+\frac{3adx}{15acd}+\frac{15acn}{15acd}=\frac{5bcd+3adx+15acn}{15acd}$, the Sum.

From $\frac{a}{c}$ take $\frac{b}{x}$, then $\frac{ax}{cx}-\frac{bc}{cx}=\frac{ax-bc}{cx}$ the Difference; from $\frac{ns}{d}$ take $\frac{c-b}{m}$, then $\frac{mns}{dm}-\frac{bd-cd}{dm}=\frac{mns+bd-cd}{dm}$, the Difference.

CASE

CASE VII.

To multiply a Fraction by a an Integer or whole Quantity.

RULE.

Multiply the Numerator of the Fraction by the given Multiplier.

Thus the Product of $\frac{c}{a}$ multiplied by x, is $\frac{c \times x}{a}$, or $\frac{cx}{a}$; that of $\frac{3c}{\sqrt{a^2+x^2}}$ multiplied by $2n$, is $\frac{6cn}{\sqrt{a^2+x^2}}$; that of $\frac{x^2-4a^2}{d}$ multiplied by $3b$, is $\frac{3bx^2-12a^2b}{d}$; and that of $\frac{c+x}{\sqrt{ax}}$ multiplied by $\sqrt{ax}$, is $\frac{\overline{c+x} \times \sqrt{ax}}{\sqrt{ax}} =$ $c+x$, a whole Quantity; for here the Multiplier is equal to the Denominator of the Fraction.

CASE VIII.

To multiply one Fraction by another.

RULE.

Multiply their Numerators together for a new Numerator, and the Denominators together for a new Denominator.

Thus $\frac{a}{c} \times \frac{b}{x} = \frac{ab}{cx}$ Product; $\frac{a+b}{c} \times \frac{a-b}{d} = \frac{a^2-b^2}{cd}$, the Product; and $\frac{\sqrt{ac}}{a-s} \times \frac{\sqrt{ac}}{d} = \frac{ac}{ad-ds}$ the Product.

CASE IX.

To divide a Fraction by an Integer.

RULE.

Divide the Numerator of the Fraction by the Integer, if the Numerator is a Multiple of it; if it is not multiply the Denominator by the Integer.

Thus $\frac{cx}{a}$ divided by x, gives $\frac{cx \div x}{a}$, or $\frac{c}{a}$; $\frac{6cn}{\sqrt{a^2+x^2}}$ divided by $2n$, gives $\frac{3c}{\sqrt{a^2+x^2}}$; and $\frac{x}{6}$ divided by 3, the Quotient is $\frac{x}{18}$; divide $\frac{a}{b}$ by c, the Quotient is $\frac{a}{bc}$; $\frac{an}{b+c}$ divided by ab, gives $\frac{an}{ab^2+abc}$, or $\frac{n}{b^2+bc}$ and $\frac{12bxy}{\sqrt{a^2-x^2}}$ divided by $4bc^2$, gives $\frac{3xy}{c^2\sqrt{a^2-x^2}}$.

CASE X.

To divide one Fraction by another.

RULE.

Multiply the Numerator of the Dividend by the Denominator of the Divisor, their Product shall give the Numerator of the Quotient. Then multiply the Denominator of the Dividend by the Numerator of the Divisor, and the Product shall give the Denominator.

Thus $\frac{c}{a}\big)\,\frac{x}{d}\,\big(\frac{ax}{cd}$ the Quotient; also $\frac{a}{a-b}\big)\,\frac{a-b}{b+c}\,\big(\frac{a^2-2ab+b^2}{ab+ac}$, the Quotient; and $\frac{\sqrt{ca}}{a-s}\big)\,\frac{ac}{ad-ds}$

$$\frac{\overline{a-s} \times ac}{\overline{ad-ds} \times \sqrt{ac}} = \frac{\overline{a-s} \times ac}{\overline{a-s} \times d\sqrt{ac}} = \frac{ac}{d\sqrt{ac}} = \frac{ac\sqrt{ac}}{acd}$$
$$= \frac{\sqrt{ac}}{d}.$$

Here the Quotient comes out $\frac{\sqrt{ac}}{d}$, and is the Multiplier in the last Example of multiplying Fractions, which proves the Truth of that Rule. And the Term $\frac{ac}{d\sqrt{ac}}$ in this Operation, was reduced to $\frac{ac\sqrt{ac}}{acd}$, or $\frac{\sqrt{ac}}{d}$, by multiplying both its Numerator and Denominator by $\sqrt{ac}$: Whence it appears, that multiplying the Numerator and Denominator of a Fraction by the same Quantity does not alter its Value, if it did, the Quotient here brought out could not have been $\left(\frac{\sqrt{ac}}{d}\right)$ a Proof to the preceding Rule, as we find it is, therefore reducing Fractions to a common Denominator, cannot alter their respective Values; since the Numerator and Denominator of each Fraction (separately taken) are multiplied by the same Quantity; thus the Fractions $\frac{a}{c}$ and $\frac{b}{x}$, when brought to a common Denominator will be $\frac{ax}{cx}$ and $\frac{bc}{cx}$, where it is obvious that the Numerator and Denominator of the first Fraction are multiplied by x; the second Fraction is multiplied in the same Manner by c; and it is plain that these Fractions $\frac{ax}{cx}$ and $\frac{bc}{cx}$, are respectively equal to $\frac{a}{c}$ and $\frac{b}{x}$: Hence it is evident that the Sum and Difference of the new Numerators, divided by their common Denominator, must be respectively equal to the Sum and Differ-

ence of those Fractions divided separately, both before and after such a Reduction, for the Sum of the Whole, divided by the common Denominator, must be equal to the Sum of all the Parts divided by the same Divisor, and the Difference of those Fractions after Reduction is the same as before.

To illustrate this by Numbers, let $a=3$, $c=4$, $b=1$, and $x=2$; then the Sum of those Fractions will be $\frac{a}{c}+\frac{b}{x}=\frac{3}{4}+\frac{1}{2}=1\frac{1}{4}$, and their Difference $\frac{a}{c}-\frac{b}{x}=\frac{3}{4}-\frac{1}{2}=\frac{1}{4}$.

So likewise $\frac{ax}{cx}+\frac{bc}{cx}=\frac{ax+bc}{cx}=\frac{3\times2+1\times4}{4\times2}=\frac{6+4}{8}=\frac{10}{8}=\frac{5}{4}=1\frac{1}{4}$, their Sum, and $\frac{ax-bc}{cx}=\frac{6-4}{8}=\frac{2}{8}=\frac{1}{4}$, their Difference, the very same as before.

The Rule for dividing a Fraction by an Integer, by multiplying its Denominator into the Integer (when it is not contained in its Numerator), is founded on this obvious Principle, that any Quantity being divided, and the Quotient thence arising being divided by another Divisor, &c. the last Quotient shall be always equal to the Quotient of the same Quantity, divided at once by the Product of all those Divisors: For Instance, if any Number were divided by 4, and if the Quotient thence arising were divided by 6, it is evident that the last Quotient would be equal to the Quotient of the same Number divided by 24, the Product of the two Divisors, and so of others.

CASE XI.

A whole Quantity (ab) divided by a Fraction, thus $\frac{ab}{\frac{c+x}{d}}$, (by multiplying (ab) its Numerator by d) may

be

be reduced to this Form $\frac{abd}{c+x}$, and is the Quotient of ab divided by $\frac{c+x}{d}$; so that to divide a whole Quantity by a Fraction, you multiply the whole Quantity by the Denominator of the Fraction, and write the Numerator of the given Fraction under the Product.

CASE XII.

If it is required to reduce a given Fraction to a Fraction equal to it that shall have a given Denominator, you must multiply the Numerator by the given Denominator, and under the proposed Fraction, so multiplied, write the given Denominator: Thus if it were required to reduce $\frac{b}{a}$ to an equal Fraction whose Denominator shall be c, then $\frac{bc \div a}{c}$, is the Fraction required.

Or putting $n = \frac{b}{a}$, you will have $\frac{cn}{c}$ for the Fraction sought.

N. B. Writing one equal Quantity instead of another (as above) is called Substitution, it is sometimes used to facilitate the Resolution of Equations, and is highly necessary in many Cases.

CASE XIII.

Fractions connected together by the Word of, where one Fraction expresses a given part of another, are reduced to simple Fractions, by multiplying their Numerators together for a new Numerator, and the Denominators together for a new Denominator.

Thus $\frac{3}{4}$ of $\frac{2a}{5}$ is $\frac{6a}{20}$, or $\frac{3a}{10}$; and the $\frac{a}{c}$ part of $\frac{b}{d}$ is $\frac{ab}{cd}$.

SECTION VII.

OF INFINITE SERIES.

When it happens in Division that the Divisor is not exactly contained in the Dividend, the Operation may be continued without End; and the Quotient will in that Case be an infinite Series of Terms.

Thus if it were required to divide 1 by $1-x$, you will find the Quotient to be $1+x+x^2+x^3+x^4+$, &c.

The Operation is thus:

$$
\begin{array}{llllll}
1-x) & 1 & & & & (1+x+x^2+x^3+x^4+, \&c. = \dfrac{1}{1-x} \\
 & 1-x & & & & \\
\hline
 & +x & & & & \\
 & +x-x^2 & & & & \\
\hline
 & & +x^2 & & & \\
 & & +x^2-x^3 & & & \\
\hline
 & & & +x^3 & & \\
 & & & +x^3-x^4 & & \\
\hline
 & & & & +x^4 & \\
 & & & & +x^4-x^5 & \\
\hline
 & & & & & +x^5, \&c. \\
\hline
\end{array}
$$

Here it is easy to see in what Order the succeeding Terms of the Quotient will arise; this is called discovering the Law of the Series, by which Means, without any more Division, the Quotient may be continued as far as you please.

By

By proceeding in the very same Manner you will find

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \&c.$$

Here the Exponent of x also increases continually by unity from the second Term of the Quotient; but the Signs of these Terms are alternately + and —.

Let it be required to divide c by $a+x$.

See the Operation.

$$\begin{array}{l}
a+x\,)\; c \quad \left(\frac{c}{a} - \frac{cx}{a^2} + \frac{cx^2}{a^3} - \frac{cx^3}{a^4} + \&c = \frac{c}{a+x}\right. \\
\quad c + \frac{cx}{a} \\
\hline
\quad\quad - \frac{cx}{a} \\
\quad\quad - \frac{cx}{a} \quad \frac{cx^2}{a^2} \\
\hline
\quad\quad\quad + \frac{cx^2}{a^2} \\
\quad\quad\quad + \frac{cx^2}{a^2} + \frac{cx^3}{a^3} \\
\hline
\quad\quad\quad\quad - \frac{cx^3}{a^3} \\
\quad\quad\quad\quad - \frac{cx^3}{a^3} - \frac{cx^4}{a^4} \\
\hline
\quad\quad\quad\quad\quad + \frac{cx^4}{a^4}, \&c.
\end{array}$$

 Here

Here, I divide c by a (the first Term of the Divisor) and the Quotient is $\frac{c}{a}$, by which I multiply $a+x$ the whole Divisor, and the Product is $\frac{ac}{a}+\frac{cx}{a}$ or $c+\frac{cx}{a}$, which subtracted from the Dividend c, there remains $-\frac{cx}{a}$, this Remainder being divided by a (the first Term of the Divisor) gives $-\frac{cx}{a^2}$, for the second Term of the Quotient, by which I also multiply $a+x$ the Divisor, and the Product is $-\frac{acx}{a^2}-\frac{cx^2}{a^3}$, or $-\frac{cx}{a}-\frac{cx^2}{a^2}$, which being taken from $-\frac{cx}{a}$, leaves $+\frac{cx^2}{a^2}$.

The Rest of the Quotient is found in the same Manner, and having obtained four Terms thereof, as above, the Law of Continuation becomes obvious; but a few of the first Terms of the Series are generally near enough the Truth for most Purposes; and in order to have a true Series, the greatest Term of the Divisor (and of the Dividend if it consists of more than one Term) must always stand first. Thus in the last Example; if x is greater than a, then x must be the first Term of the Divisor, and the Quotient will be $\frac{c}{x+a}=\frac{c}{x}-\frac{ac}{x^2}+\frac{a^2c}{x^3}-\frac{a^3c}{x^4}+$, &c. the true Series; but if x is less than a, then this Series is false, and the further it is continued, the more it will diverge from the truth:

For let $a=2$, $c=1$, and $x=1$; then if the Division be performed with a, as the first Term of the Divisor, you will have $\frac{c}{a+x}=\frac{1}{2+1}=\frac{1}{2}-\frac{1}{4}+\frac{1}{8}-\frac{1}{16}$, $+$ &c. $=\frac{1}{3}$.

But

But if x be placed first in the Divisor, then will

$$\frac{c}{x+a}\left(=\frac{1}{3}\right)=\frac{1}{1+2}=1-2+4-8+16-, \text{\&c.}$$

Now it is obvious, that the first Series continually converges to the Truth, for the first Term thereof, viz. $\frac{1}{2}$, exceeds the Truth by ($\frac{1}{2}-\frac{1}{3}$, or) $\frac{1}{6}$; two Terms are deficient by $\frac{1}{12}$; three Terms exceed it by $\frac{1}{24}$; four Terms are deficient by $\frac{1}{48}$; five Terms will exceed the Truth by $\frac{1}{96}$, &c. So that each succeeding Term of the Series brings the Quotient continually nearer and nearer to the Truth by one Half of its last preceding Difference; and consequently the Series will approximate to the Truth nearer than any assigned Number or Quantity whatever; and it will converge so much the swifter as the Divisor is greater than the Dividend.

But the second Series perpetually diverges from the Truth; for the first Term of the Quotient exceeds the Truth by $1-\frac{1}{3}$, or $\frac{2}{3}$; two Terms thereof are deficient by $\frac{4}{3}$; three Terms exceed it by $\frac{8}{3}$; four Terms are deficient by $\frac{16}{3}$; five Terms exceed the Truth by $\frac{32}{3}$, &c. which shews the Absurdity of this Series. For the same Reason x must be less than Unity in the second Example; if x were there equal to Unity, then the Quotient would be alternately 1, and nothing, instead of $\frac{1}{2}$; and it is evident that x is less than Unity in the first Example, otherwise the Quotient could not have been affirmative, for if x be greater than Unity; then $1-x$, the Divisor is negative, and unlike Signs in Division give negative Quotients. From the Whole of which it appears that the greatest Term of the Divisor must always stand first.

SECTION VIII.

OF INVOLUTION.

CASE I.

INVOLUTION is a continual Multiplication of a Quantity into itſelf, and the Products thence ariſing are called the Powers of that Quantity, and the Quantity itſelf is called the Root; thus as $a\times a=aa$, is the ſecond Power of the Root a, ſo $ab\times ab=aabb$, is the ſecond, and $ab\times ab\times ab=aaabbb$, is the third Power of the Root ab.

CASE II.

If the Quantities have Co-efficients, they muſt be involved with the Quantities; thus $2x\times 2x\times 2x=8xxx$, is the third Power of $2x$, $3xy\times 3xy\times 3xy\times 3xy=81xxxxyyyy$, is the fourth Power of $3xy$, &c.

But Quantities may be involved by multiplying their Exponents by that of the Power required (as has been ſhewn in Notation.) Thus the third Power of x^2 is $x^{2\times 3}$, or x^6; the fourth Power of $2a^3b^2$ is $2^4\times a^{12}b^8$, or $16a^{12}b^8$; the m^{th} Power of $a^n b$ is $a^{mn} b^m$; the ſecond Power of $\overline{ax}|^{\frac{1}{2}}$ is $\overline{ax}|^{\frac{1}{2}\times 2}$, or $\overline{ax}|^{\frac{2}{2}}$, that is, ax, the n^{th} Power of $ax^{\frac{1}{n}}$ is $ax^{\frac{n}{n}}$, or ax, and the m^{th} Power of $\overline{a^2+x^2}|^{\frac{n}{3m}}$ is $\overline{a^2+x^2}|^{\frac{mn}{3m}}$, or $\overline{a^2+x^2}|^{\frac{n}{3}}$, namely the n^{th} Power of the Cube Root of a^2+x^2.

CASE III.

All the odd Powers raiſed from a negative Root are negative, and all the even Powers are poſitive; thus the ſecond

cond Power of $-a$ is $-a\times-a=+a^2$ (by the Rule for the Signs in Multiplication) the third Power of $-a$ is $+a^2\times-a=-a^3$, the fourth Power is $-a^3\times-a=+a^4$, the fifth Power of $-a$ is $+a^4\times-a=-a^5$, &c, &c.

CASE IV.

A Fraction is involved by raising both its Numerator and Denominator to the Power proposed; thus the second Power of $\frac{a}{c}$ is $\frac{a^2}{c^2}$; the third Power of $\frac{1}{a}$ is $\frac{1}{a^3}$ the fourth Power of $\frac{2ab^{\frac{1}{2}}}{x^{\frac{1}{4}}}$ is $\frac{16a^4b^2}{x}$ and the fifth Power of $\frac{\overline{a+x}|^{\frac{3}{5}}}{\overline{a-c}|^{\frac{1}{2}}}$ is $\frac{\overline{a+x}|^3}{\overline{a-c}|^{\frac{5}{2}}}$.

CASE V.

Negative Powers, as well as positive, are multiplied by adding, and divided by subtracting their Exponents; thus the Product of x^{-2} multiplied by x^{-1}, is $x^{-2-1}=x^{-3}$ $=\frac{1}{x^3}$, also $x^{-5}\times x^4=x^{-5+4}=x^{-1}=\frac{1}{x}$, and x^2 $\times x^{-2}=x^{2-2}=x^0=1$; for here the positive and negative Powers destroy one another, and produce only x^0, or Unity, in the Product, which is no Power of x at all.

Moreover it has been proved in Division that the Factor x^{-2} is equal to $\frac{1}{x^2}$, so that the negative Powers of x are positive Powers of $\frac{1}{x^2}$, and $\frac{1}{x^2}$, multiplied by x^2, gives $\frac{1\times x^2}{x^2}=\frac{x^2}{x^2}=1$, the very same as before; hence it appears that if any positive Power of x be multiplied by a negative Power of x of an equal Exponent, the Product will be Unity.

Now

Now it will be easy to perceive how any Quantity placed in the Denominator of a Fraction may be transposed to the Numerator, if the Sign of its Exponent be changed.

For since $x^{-3} = \frac{1}{x^3}$, it is evident that $\frac{b}{ax^3} = \frac{b}{a} \times \frac{1}{x^3} = \frac{b}{a} \times x^{-3} = \frac{b \times x^{-3}}{a} = \frac{bx^{-3}}{a}$, and $\frac{b}{ax^{-3}} = \frac{b}{a \times \frac{1}{x^3}} = \frac{b}{\frac{a \times 1}{x^3}} = \frac{b}{\frac{a}{x^3}} = \frac{bx^3}{a}$. (See Case II. in Reduction of Fractions.)

In the same Manner $\sqrt{\frac{1}{x}} = \frac{1}{x^{\frac{1}{2}}} = \frac{1 \times x^{-\frac{1}{2}}}{1} = x^{-\frac{1}{2}}$; also $\frac{ab}{x^2y^3} = abx^{-2}y^{-3}$; $\frac{1}{a^2+x^2} = \overline{a^2+x^2}^{-1}$; $\frac{1}{\overline{a^2+x^2}|^2} = \overline{a^2+x^2}^{-2}$, and $\frac{1}{\overline{a^2+x^2}|^3} = \overline{a^2+x^2}^{-3}$, &c.

In like Manner $\sqrt{\frac{c}{a^2-x^2}} = c \times \overline{a^2-x^2}^{-\frac{1}{2}}$ and $\frac{b}{\overline{ax+y^2}|^{\frac{n}{m}}} = b \times \overline{ax+y^2}^{-\frac{n}{m}}$.

CASE VI.

Quantities compounded of several Terms, are involved by an actual Multiplication of all their Parts; thus the Powers of any Binomial $a+b$ are found by a continual Multiplication of it by itself, as follows:

$a+b$

$a+b$, the Root, or first Power.
$\times a+b$

a^2+ab
$+ab+b^2$

$a^2+2ab+b^2$, the second Power.
$\times a+b$

$a^3+2a^2b+ab^2$
$+a^2b+2ab^2+b^3$

$a^3+3a^2b+3ab^2+b^3$, the third Power.
$\times a+b$

$a^4+3a^3b+3a^2b^2+ab^3$
$a^3b+3a^2b^2+3ab^3+b^4$

$a^4+4a^3b+6a^2b^2+4ab^3+b^4$ the fourth Power.
$\times a+b$

$a^5+4a^4b+6a^3b^2+4a^2b^3+ab^4$
$a^4b+4a^3b^2+6a^2b^3+4ab^4+b^5$

$a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$, the fifth Power, &c.

The Powers of $a-b$ are raised in the same Manner, regard being had to the Signs.—See the Operation.

$a-b$
$\times a-b$

a^2-ab
$-ab+b^2$

$a^2-2ab+b^2$ the second Power
$\times a-b$

$$\begin{array}{l} a^3-2a^2b+ab^2 \\ \quad -a^2b+2ab^2-b^3 \\ \hline a^3-3a^2b+3ab^2-b^3 \text{ the third Power} \\ a-b \\ \hline a^4-3a^3b+3a^2b^2-ab^3 \\ \quad -a^3b+3a^2b^2-3ab^3+b^4 \\ \hline a^4-4a^3b+6a^2b^2-4ab^3+b^4 \text{ the fourth Power.} \\ \times a-b \\ \hline a^5-4a^4b+6a^3b^2-4a^2b^3+ab^4 \\ \quad -a^4b+4a^3b^2-6a^2b^3+4ab^4-b^5 \\ \hline a^5-5a^4b+10a^3b^2-10a^2b^3+5ab^4-b^5 \text{ the fifth Power, \&c.} \\ \hline \end{array}$$

Here you may obferve, that the feveral Powers of this Binomial $a-b$ confift of the fame Terms as thofe of the foregoing one $a+b$; the Difference being only in the Signs of thofe Terms in which the Exponent of b is an odd Number, as b, b^3, b^5, &c. thefe are all negative, but all the even Powers of b, as b^2, b^4, &c. are pofitive. And it is to be obferved, that in the firft and laft Terms of any Power of $a \mp b$, the Quantities a and b will each have Unity for their Co-efficients, and each of their Exponents will be equal to that of the Power required.

The Quantity b commences, in its firft Power, in the fecond Term of any Power of $a \mp b$, and its Exponent increafes by Unity in each fucceeding Term, as that of a decreafes; fo that the Sum of their Exponents is ever the fame, and is always equal to the Exponent of the Power propofed.

Thus in the fifth Powers of the preceding Examples the Exponent of a decreafes in this order, 5, 4, 3, 2, 1, 0, and thofe of b increafes in the contrary order 0, 1, 2, 3, 4, 5, and the Sum of their Exponents in any Term is 5.

For the Exponent of a^5 the firft Term of thofe Powers is 5, the Sum of the Exponents of a and b in $5a^4b$, the fecond

second Term, is $4+1=5$, in the third Term $10a^3b^2$, their Sum is $3+2=5$, &c, &c.

CASE VII.

To find the Co-efficient of any proposed Term, that of its preceding Term being known.

RULE.

Add Unity to the Exponent of a in the Term proposed, divide that Sum by the Exponent of b in the same Term, and multiply the Quotient by the Co-efficient of the preceding Term; thus to find the Co-efficients of the Terms of the fifth Power of $a+b$.

The Co-efficient of a^5 the first Term being Unity, that of the second Term (by the Rule) will be $1 \times \frac{4+1}{1}$, or $4+1$, or 5, that of the third Term will be $5 \times \frac{3+1}{2}$, or $5 \times \frac{4}{2}=10$, that of the fourth Term will be $10 \times \frac{2+1}{3}=10$; the Rest of the Co-efficients will be found in the same Manner to be 5. 1. and since the Exponents have such a Relation to the Co-efficients, it will be easy to perceive how the Powers of a Binomial may be expressed in a general Manner, without the Trouble or those tedious Multiplications required in the foregoing Operations, which is thus:

Let n denote any Number at pleasure; then the first Term of the n^{th} Power of $a+b$, will be represented by a^n, and because the Exponent of a decreases continually by Unity in every following Term of the Series, therefore the second Term will be denoted by $a^{n-1}b$, the third Term by $a^{n-2}b^2$, the fourth Term by $a^{n-3}b^3$, &c. so that the Terms of the general Theorem without their Co-efficients will stand thus, $a^n+a^{n-1}b+a^{n-2}b^2+a^{n-3}b^3$, &c. continued till the Exponent of b becomes equal to n, for then

then the Series must end, because the Exponent of a will, in that Case, be nothing, as shall be shewn further on.

The Co-efficient of the first Term being Unity, and that of the second Term being always equal to (n) the Exponent of the Power, the Rest of the Co-efficients may be readily found by the preceding

RULE.

Thus, by adding Unity to $n-2$ the Exponent of a in the third Term, then dividing $n-2+1$, or $n-1$, the Sum, by 2, the Exponent of b in the same Term, and multiplying $\frac{n-1}{2}$ the Quotient by n, the Co-efficient of the second Term, gives $n \times \frac{n-1}{2}$, for the Co-efficient of the third Term: The Exponent of a, more 1, in the fourth Term, divided by 3, the Exponent of b in the same Term, gives $\frac{n-3+1}{3}$, or $\frac{n-2}{3}$, this multiplied by $n \times \frac{n-1}{2}$, the Co-efficient of the third Term, gives $n \times \frac{n-1}{2} \times \frac{n-2}{3}$, for the Co-efficient of the fourth Term; hence the Law of Continuation is evident, and the Co-efficients expressed in a general Manner will stand thus, $1, n, n \times \frac{n-1}{2}, n \times \frac{n-1}{2} \times \frac{n-2}{3}, n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}, n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5}$ &c. continued till you have one Co-efficient more than there are Units in n, because the Terms in any Power of $a+b$, are always one more than the Units in the Exponent.

And

And the Terms of the general Theorem with their Co-efficients will become $a^n + na^{n-1}.b + n \times \frac{n-1}{2} a^{n-2} b^2 + n \times \frac{n-1}{2} \times \frac{n-2}{3} a^{n-3} b^3$, &c. Therefore $\overline{a+b}|^n = a^n + na^{n-1} b + n \times \frac{n-1}{2} a^{n-2} b^2 + n \times \frac{n-3}{3} a^{n-3} b^3$, &c. which is the general Theorem for raising a Quantity consisting of two Terms to any Power n.

And the n^{th} Power of $a-b$ will be expressed in the very same Manner, only the Signs of the second, fourth, sixth, &c. Terms where the odd Powers of b are involved, must be negative.

To shew the Use of this Theorem, let it be required to raise $a+b$ to the third Power. Here n, the Exponent of the proposed Power being 3, the first Term a^n, of the Theorem, will become a^3, the second Term $na^{n-1} b$, will be $3a^{3-1} b$, or $3a^2b$, the third Term $n \times \frac{n-1}{2} a^{n-2} b^2$, will be $3 \times \frac{3-1}{2} a^{3-2} b^2$, or $3 \times \frac{2}{2} a^1 b^2$, that is, $3ab^2$, the fourth Term, viz. $n \times \frac{n-1}{2} \times \frac{n-2}{3} a^{n-3} b^3$, will become $3 \times \frac{3-1}{2} \times \frac{3-2}{3} a^{3-3} b^3 = 3 \times \frac{2}{2} \times \frac{1}{3} a^0 b^3 = \frac{6}{6} a^0 b^3 = b^3$.

In this Term you see the Co-efficient is equal to Unity, the positive and negative Powers of a destroy each other, and the Exponent of b is become equal to 3, the Units in n, which shews that the Series is ended.

Hence the third Power of $a+b$, is found to be $a^3 + 3a^2b + 3ab^2 + b^3$.

Again, let it be required to raise $a+b$ to the sixth Power, in which Case the Exponent n, being 6, the first Term a^n of the general Theorem is equal to a^6, the second Term $na^{n-1} b = 6a^5b$, the third Term $n \times \frac{n-1}{2} a^{n-2} b^2 = 6 \times$

$\frac{6-1}{2}$

$\frac{6-1}{2} a^{6-2} b^2 = 15a^4 b^2$, the fourth Term $n \times \frac{n-1}{2} \times \frac{n-2}{3} a^{n-3} b^3 = 6 \times \frac{5}{2} \times \frac{4}{3} a^3 b^3 = 20a^3b^3$; proceeding on in this Manner, you will find the other three Terms to be $15a^2b^4$, $6ab^5$, b^6, therefore $\overline{a+b}|^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$.

By these and the preceding Operations, you may observe that the Co-efficients increase to half the Number of Terms in the odd Powers of $a+b$, and to the middle Term in the even Powers of $a+b$; and then they decrease again in the same Order, therefore you need not find the Co-efficients in this Manner, further than those Terms, since from those the Rest will be given.

By the foregoing Theorem you will find $\overline{a-b}|^{r+1} = a^{r+1} - \overline{r+1} \times a^r b + \overline{r+1} \times \frac{r}{2} a^{r-1} b^2 - \overline{r+1} \times \frac{r}{2} \times \frac{r-1}{3} a^{r-2} b^3$, +, &c.

Here you see the Exponent of the Power is greater by 1, than the Units in r; and if you write n instead of $r+1$, the Exponents and Co-efficients of the Terms will be expressed the same as before, Regard being had to the Signs of those Terms in which the odd Powers of b are involved, as above.

If a Trinomial, as $a+b+c$, be involved, you will find the second Power to be $a^2 + 2ab + b^2 + 2ac + 2bc + c^2$; and the third Power $a^3 + 3a^2b + 3ab^2 + b^3 + 3a^2c + 6abc + 3b^2c + 3ac^2 + 3bc^2 + c^3$. Now it is obvious, that the Sum of the first three Terms in the second Power of $a+b+c$, is the Square of $a+b$; and the three last Terms are equal to $2c \times \overline{a+b} + c^2$; therefore $\overline{a+b+c}|^2 = \overline{a+b}|^2 + 2c \times \overline{a+b} + c^2 = a^2 + 2ab + b^2 + 2ac + 2bc + c^2$. Again, it is plain, that the first four Terms of the third Power are equal to $\overline{a+b}|^3$, the next three following Terms are equal to $3c \times \overline{a+b}|^2$, and the three last Terms are equal to $3c^2 \times \overline{a+b} + c^3$, and consequently $\overline{a+b+c}|^3 = \overline{a+b}|^3 + 3c \times \overline{a+b}|^2 +$

$3c^2 \times$

$+3c^2\times\overline{a+b}+c^3=a^3+3a^2b+3ab^2+b^3+3a^2c+6abc+3bc^2$ $+3ac^2+3bc^2+c^3$.

Hence you may obſerve, that the Terms of the Square and Cube of any Trinomial may be known from thoſe of a Binomial, without actual Involution.

And by proceeding on in this Manner you may find Theorems for involving Quantities conſiſting of more Terms, in which Theorems, the ſeveral Terms of the Powers may be had by Inſpection, as in thoſe above.

SECTION IX.

OF EVOLUTION.

CASE I.

EVOLUTION is the Reverſe of Involution, and conſiſts in reſolving Powers into their Roots, which Operation is likewiſe called Extraction; and as the Involution of Quantities was performed by multiplying their Exponents, ſo their Roots may be extracted (as has been ſhewn in Notation) by dividing their Exponents by the Number that denominates the Root required.

Thus the ſquare Root of a^4 is $a^{\frac{4}{2}}$, or a, the Square Root of $a^4b^6c^8$ is $a^2b^3c^4$, the ſquare Root of $\overline{ax}|^{\frac{6}{5}}$ is $\overline{ax}|^{\frac{3}{5}}$, and the ſquare Root of $\overline{ax}|^{\frac{1}{n}}$ is $\overline{ax}|^{\frac{1}{2n}}$. In the firſt three Quantities you ſee the Exponents are each divided by 2; but in the laſt the Denominator of the Exponent is multiplied by 2, agreeable to the Rule already demonſtrated, for dividing a Fraction by multiplying its Denominator by the Diviſor, when it is not contained in the Numerator.

Moreover, the Cube Root of a^3 is $a^{\frac{3}{3}}$, or a, the Cube Root of $8\,a^3b^6$ is $2ab^2$, the Biquadratic Root of $\overline{ax}|^{\frac{2}{3}}$ is $\overline{ax}|^{\frac{2}{12}}$ or $\overline{ax}|^{\frac{1}{6}}$, and the n^{th} Root of $\overline{a^2+x^2}|^{\frac{n}{m}}$ is $\overline{a^2+x^2}|^{\frac{1}{m}}$.

CASE II.

From what has been said with respect to the Signs in Involution (Case III.) it appears, that any Power which has an affirmative Sign may have an affirmative or negative Root, when the Exponent of that Root is an even Number. Thus the square Root of $+a^2$ may be $+a$, or $-a$, because $+a\times+a=+a^2$, and also $-a\times-a=+a^2$. From thence it likewise follows, that no Root whose Exponent is an even Number can be found for a Power with a negative Sign.

Thus the square Root cannot be extracted from $-a^2$, because there is no Root, positive or negative, which will when multiplied into itself, produce $-a^2$, and therefore such Roots of negative Quantities are termed impossible or imaginary, and yet they will sometimes come into Use.

But if the Power to be extracted be denominated by an odd Number, then the Sign of the Root will be the same as that of the Power.

Thus the Cube Root of $-a^3$ is $-a$; of $+a^3$ it is $+a$; and the sursolid Root of $-a^5$ is $-a$.

CASE III.

To find the Root of a Fraction.

RULE.

Extract the Root of its Numerator and Denominator:

Thus the Square Root of $\frac{a^2}{c^2}$ is $\frac{a}{c}$, the Cube Root of

$\frac{1}{a^3}$

$\frac{1}{a^3}$ is $\frac{1}{a}$, the Biquadratic Root of $\frac{16a^4b^2}{x}$ is $\frac{2ab^{\frac{1}{2}}}{x^{\frac{1}{4}}}$ and the surfolid Root of $\frac{\overline{b+x}|^3}{\overline{a-c}|^{\frac{5}{2}}}$ is $\frac{\overline{b+x}|^{\frac{3}{5}}}{\overline{a-c}|^{\frac{1}{2}}}$.

CASE IV.

Evolution of compound Quantities is performed by the following

RULE.

First, range their several Terms according to the Dimensions of the Letters in them, as in Division, and by twice the Root of the first Term divide the second Term of the given Quantity, or by three Times its Square, or four Times its Cube, &c. according as the Root to be extracted is a Square, Cube, or Biquadratic one, &c. and the Quotient which arises by that simple Divisor shall be the second Term of the Root; then subtract the square, cube, or fourth Power, &c. of those two Terms in the Root from the given Quantity, according as the Root to be subtracted is a Square, Cube, or Biquadratic one, &c. and if there be a Remainder divide its first Term by the same simple Divisor that you divided the second Term of the given Quantity by, and the Quotient shall be the third Term of the Root, &c.

The following Examples will make this very easy.

EXAMPLE I.

Let it be required to extract the Square Root of $a^2-4ax+4x^2+2ay-4xy+y^2$.

Here dividing $-4ax$ the second Term, by $2a$, twice the Square Root of a^2, the first Term, gives $-2x$ for the second Term of the Root, then subtracting the Square of $a-2x$ (the two first Terms of the Root) from the given Quantity, the Remainder is $2ay-4xy+y^2$, and dividing $2ay$, the first Term of this Remainder, by $2a$, twice

the first Term of the Root, gives y for the third Term of the Root, and by subtracting the Square of $a-2x+y$ (which is $a^2-4ax+4x^2+2ay-4xy+y^2$) from the given Quantity, there remains nothing, and therefore $a-2x+y$ is the Square Root required.

But besides the foregoing general Rule for extracting the Roots of all Powers, there is another by which the Square Root of a Quantity may be more readily obtained; which is thus: Find the Square Root of the first Term a^2, which gives a for the first Term of the Root, then subtract the Square of a, the first Term of the Root from the proposed Quantity, and divide the first Term of the Remainder $-4ax+4x^2+2ay-4xy+y^2$, by $2a$, twice the first Term of the Root, and the Quotient $-2x$ is the second Term of the Root; add this second Term to the double of the first, and multiply their Sum $2a-2x$ by the second Term $-2x$; subtract the Product $-4ax+4x^2$ from the aforesaid Remainder, and divide the first Term of this Remainder $2ay-4xy+y^2$ by $2a$, twice the first Term of the Root, and add the Quotient y, to the double of the two first Terms of the Root for the next compound Divisor, &c. See the Operation at large.

$$
\begin{array}{rl}
 & a^2-4ax+4x^2+2ay-4xy+y^2\ (a-2x+y \text{ Root.} \\
 & a^2 \\
\hline
2a-2x\ \big) & -4ax+4x^2+2ay-4xy+y^2 \\
\times -2x\ \big) & -4ax+4x^2 \\
\hline
2a-4x+y\ \big) & +2ay-4xy+y^2 \\
\times y\ \big) & \quad 2ay-4xy+y^2 \\
\hline
 & \quad 0 \quad 0 \quad 0 \\
\hline
\end{array}
$$

In the same Manner the Root of a Surd may be extracted, and expressed by an infinite Series: Thus you will find the Square Root of a^2+x^2, to be $a+\frac{x^2}{2a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^5}-$ &c.

For

For here the Square Root of the first Term a^2, is a, the first Term of the Root, which being squared and taken from the given Surd a^2+x^2, leaves x^2, this Remainder divided by $2a$, twice the first Term of the Root, gives $\frac{x^2}{2a}$ for the second Term of the Root, which added to $2a$, gives $2a+\frac{x}{2a}$ for the first compound Divisor, which being multiplied by $\frac{x^2}{2a}$, and the Product $(x^2+\frac{x^4}{4a^2})$ taken from the first Remainder x^2, leaves $-\frac{x^4}{4a^2}$, this Remainder divided by $2a$, gives $-\frac{x^4}{8a^3}$ for the third Term of the Root, which must be added to the double of $(a+\frac{x^2}{2a})$ the two first Terms of the Root for the next compound Divisor, &c.—See the Work.

$$
\begin{array}{lrl}
 & a^2+x^2 & \Big(\, a+\frac{x^2}{2a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^5}-\text{, \&c.} \\
 & a^2 & \\
\hline
a+\frac{x^2}{2a}\Big) & x^2 & \\
 & x^2+\frac{x^4}{4a^2} & \\
\hline
2a+\frac{x^2}{a}-\frac{x^4}{8a^3}\Big) & -\frac{x^4}{4a^2} & \\
 & -\frac{x^4}{4a^2}-\frac{x^6}{8a^4}+\frac{x^8}{64a^6} & \\
\hline
 & +\frac{x^6}{8a^4}-\frac{x^8}{64a^6}\text{, \&c.} &
\end{array}
$$

 Here

Here dividing $+\frac{x^6}{8a^4}$ the first Term of this Remainder by $2a$, gives $+\frac{x^6}{16a^5}$ for the fourth Term of the Root, and thus you may continue the Series on as far as you please.

In like Manner you will find $\sqrt{a^2-x^2} = a - \frac{x^2}{2a} - \frac{x^4}{8a^3} - \frac{x^6}{16a^5} -$, &c.

Note. In order to have a true Series, the greatest Term of the proposed Surd must be always placed first, as has been demonstrated in Section VII, so that a^2 is here supposed to be greater than x^2.

But these Surds may be more expeditiously extracted by the Binomial Theorem; for to extract any Root of a given Quantity is the same thing as to raise that Quantity to a Power whose Exponent is a Fraction; the Numerator of the Exponent shews the Power, and its Denominator expresses what kind of a Root is to be extracted.

Thus to extract the Square Root of a^2+x^2, is to raise a^2+x^2 to a Power whose Exponent is $\frac{1}{2}$, therefore if we put $n=\frac{1}{2}$, then by the Theorem we shall have $\overline{a^2+x^2}|^n =$ $a^{2n} + na^{2n-2}x^2 + n \times \frac{n-1}{2} a^{2n-4} x^4 + n \times \frac{n-1}{2} \times \frac{n-2}{3}$ $a^{2n-6}x^6$, &c. $= a^{\frac{2}{2}} + \frac{1}{2}a^{-1}x^2 + \frac{1}{2} \times -\frac{1}{4} a^{-3} x^4 + \frac{1}{2} \times -\frac{1}{4}$ $\times -\frac{1}{2} a^{-5} x^6$, &c. $= a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5}$, &c. the very same as before. In the second Term $\frac{1}{2} a^{-1} x^2$ of this Series, the Factor $a^{-1} = \frac{1}{a}$ (by Case V. of Involution) hence $\frac{1}{2} a^{-1} x^2 = \frac{1}{2} \times \frac{1}{a} \times x^2 = \frac{1x^2}{2a} = \frac{x^2}{2a}$

as before: The third Term $n\times\frac{n-1}{2}a^{2n-4}x^4=\frac{1}{2}\times$

$\frac{\frac{1}{2}-1}{2}a^{\frac{2-4}{2}}x^4=\frac{1}{2}\times\frac{-\frac{1}{2}}{2}a^{1-4}x^4=\frac{1}{2}\times-\frac{1}{4}a^{-3}x^4=-$

$\frac{x^4}{8a^3}$.

Here you see this Term comes out negative, the same as by actual Extraction; but the Co-efficient of the fourth Term is $\frac{1}{2}\times-\frac{1}{4}\times-\frac{1}{2}=+\frac{1}{16}$ affirmative, therefore the fourth Term is affirmative. Hence you may observe, that in the Extraction of Roots this Theorem brings out the Terms of the Series with their proper Signs.

Again, let it be required to extract the Cube Root of $a^3-6a^2x+12ax^2-8x^3$, and the Work will stand thus:

$$
\begin{array}{lllll}
 & a^3-6a^2x & +12ax^2-8x^3 & (a-2x, & \text{the Root required.} \\
3a^2) & -6a^2x & (-2x & & \\
 & a^3-6a^2x & +12ax^2-8x^3, & \text{the Cube of } a-2x. & \\
\hline
 & 0 \quad 0 & 0 \quad 0 & & \\
\hline
\end{array}
$$

Here you see the second Term $-6a^2x$ of the given Quantity is divided by $3a^2$, three Times the Square of a, the first Term of the Root, agreeable to the preceding Rule, and the Quotient gives $-2x$, for the second Term of the Root.

Lastly, let it be proposed to extract the surfolid Root of $\overline{a+x}|^3$ by an infinite Series. Put $n=\frac{3}{5}$, then will $\overline{a+x}|^{\frac{3}{5}}$ $=\overline{a+x}|^n=a^n+na^{n-1}x+n\times\frac{n-1}{2}a^{n-2}x+n\times$ $\frac{n-1}{2}\times\frac{n-2}{3}a^{n-3}x^3+$, &c. $=a^{\frac{3}{5}}+\frac{3}{5}a^{\frac{3}{5}-1}x+\frac{3}{5}$ $\times-\frac{1}{5}a^{\frac{3}{5}-2}x^2+\frac{3}{5}\times-\frac{1}{5}\times-\frac{7}{15}a^{\frac{3}{5}-3}x^3+$, &c. $=a^{\frac{3}{5}}$ $+\frac{3x}{5a^{\frac{2}{5}}}-\frac{3x^2}{25a^{\frac{7}{5}}}+\frac{7x^3}{125a^{\frac{12}{5}}}-$, &c. The Root required.

SECTION X.

REDUCTION OF RADICAL QUANTITIES.

CASE I.

To reduce Surds to their most simple Terms.

RULE.

Divide the Quantity under the radical Sign by its greatest square Cube, Biquadrate, &c. answering to the given Surd; prefix the Root of the Divisor to the radical Sign, and place the Quotient under it.

Thus $\sqrt{12}$ is reduced to $\sqrt{4} \times \sqrt{3} = 2\sqrt{3}$, $\sqrt{63} = \sqrt{9 \times 7} = \sqrt{9} \times \sqrt{7} = 3\sqrt{7}$ and $\sqrt{a^2x^3}$ may be reduced to $\sqrt{a^2x^2} \times \sqrt{x}$, which, by extracting the Square Root of a^2x^2, becomes $ax\sqrt{x}$.

In like Manner $\sqrt{18a^2+9a^2x} = 3a\sqrt{2+x}$ and $\sqrt{8a^2+4a^2x} = 2a\sqrt{2+x}$.

This kind of Reduction is also useful in Addition and Subtraction of Surd Quantities, where the Quotients (under the radical Sign) in each Quantity are the same.

Thus the Sum of $3a\sqrt{2+x}$ and $2a\sqrt{2+x}$ is $3a\sqrt{2+x} + 2a\sqrt{2+x} = 5a\sqrt{2+x}$; and their Difference is $3a\sqrt{2+x} - 2a\sqrt{2+x} = a\sqrt{2+x}$.

Again, $\sqrt[3]{a^5b^4} = ab\sqrt[3]{a^2b}$, and $\sqrt[n]{a^nx^{2n}} = ax\sqrt[n]{x^2}$.

In all these Surds you see the Root of the Divisor is prefixed to the radical Sign, and the Quotient of the Division is placed under it; thus the greatest cube Divisor of a^5b^4 is a^3b^3, whose Cube Root ab, is prefixed to the radical Sign, and the Quotient a^2b is placed under it. In like Manner a^nx^n divides a^nx^{2n}, and the Quotient x^2 is placed under

under the radical Sign, and the n^{th} Root of $a^n x^n$, which is ax, is prefixed to the radical Sign: And in the first Surd you see that 12, the Quantity under the radical Sign, is divided by 4, the greatest square Number that would divide it without a Remainder, and the Square Root 2 is prefixed to the radical Sign, and the Quotient 3 is placed under it.

CASE II.

A rational Quantity may be reduced to the Form of any given Surd, by raising the Quantity to the Power that is denominated by the Name of the Surd, and then setting the radical Sign over it:

Thus, $a = \sqrt[2]{a^2} = \sqrt[3]{a^3} = \sqrt[4]{a^4} = \sqrt[n]{a^n} = a^{\frac{n}{n}}$, and $2 = \sqrt[2]{4} = \sqrt[3]{8} = \sqrt[4]{16} = \sqrt[5]{32} = \sqrt[n]{2^n} = 2^{\frac{n}{n}}$.

In like Manner $\sqrt[2]{ax} = \sqrt[6]{a^3x^3} = \overline{a^3x^3}|^{\frac{1}{6}}$, and $\sqrt[3]{a} = \sqrt[6]{a^2} = \overline{a^2}|^{\frac{1}{6}}$.

This Sort of Reduction is useful in multiplying and dividing Surds under different radical Signs.

Thus $\sqrt[2]{ax}$, or $\overline{a^3x^3}|^{\frac{1}{6}}$ multiplied by $\sqrt[3]{a}$, or $\overline{a^2}|^{\frac{1}{6}}$ gives $\overline{a^3x^3}|^{\frac{1}{6}} \times \overline{a^2}|^{\frac{1}{6}}$, or $\overline{a^5x^3}|^{\frac{1}{6}}$, and the Quotient of $\overline{a^3x^3}|^{\frac{1}{6}}$ divided by $\overline{a^2}|^{\frac{1}{6}}$, gives $\overline{ax^3}|^{\frac{1}{6}}$.

But if the given Surds are of the same Quantity, and have different radical Signs, bring their Exponents to a common Denominator, then the Sum and Difference of the Exponents so reduced, shall be the Exponent of their Product and Quotient respectively.

Thus the Surds $\overline{ax}|^{\frac{m}{c}}$ and $\overline{ax}|^{\frac{n}{s}}$, when the Exponents are brought to a common Denominator, will become $\overline{ax}|^{\frac{ms}{cs}}$

and

and $\overline{ax}|^{\frac{cn}{cs}}$, then their Product is $\left(\overline{ax}|^{\frac{m}{c}} \times \overline{ax}|^{\frac{n}{s}} =\right) \overline{ax}|^{\frac{ms}{cs}}$ $\times\, \overline{ax}|^{\frac{cn}{cs}} = \overline{ax}|^{\frac{ms+cn}{cs}}$; and their Quotient is $\dfrac{\overline{ax}|^{\frac{ms}{cs}}}{\overline{ax}|^{\frac{cn}{cs}}} =$

$$\overline{ax}|^{\frac{ms}{cs} - \frac{cn}{cs}} = \overline{ax}|^{\frac{ms-cn}{cs}}.$$

It is evident that reducing Exponents to a common Denominator does not alter their respective Values, and therefore the Values of the Surds cannot be affected by such a Reduction of their Exponents, but must remain invariably the same; therefore the Sum and Difference of the Exponents of Surds, must necessarily give the Exponents of their Product and Quotient respectively, the very same as those of rational Quantities.

CASE III.

Any Binomial Surd being given, to find the Multiplier which shall produce a rational Product.

RULE.

First, if the Terms of the Binomial have not the same Indexes, reduce their Exponents to a common Denominator, or to the same radical Sign; change the Sign of one Term of the proposed Surd, and raise the two Quantities under the radical Sign to a Power whose Exponent is less by 1, than the Units in the Denominator of the Exponent of the given Surd: Prefix the common radical Sign to each Term of the Power; but reject the Co-efficients, so shall the two Quantities thus involved, be the Multiplier, which will produce a rational Product.

Here follows a general Theorem, which answers exactly to the Rule.

Binomial $\sqrt[n]{a} \pm \sqrt[n]{b}$

Multiplier $\sqrt[n]{a^{n-1}} \mp \sqrt[n]{a^{n-2}b} + \sqrt[n]{a^{n-3}b^{2}}$ $\mp \sqrt[n]{a^{n-4}b^{3}}$, + &c.

N. B. In

N. B. In this Theorem you are to obſerve, that, if both Terms of the given Binomial are poſitive, then the ſecond, fourth, ſixth, &c. Terms of the Multiplier muſt be negative; but if one Term of the Binomial Surd is negative, then all the Terms of the Multiplier muſt be affirmative; and the Number of Terms in the Multiplier will be always equal to the Units in n.

EXAMPLE I.

Let it be required to find the Surd, which multiplied by $\sqrt[3]{a}+\sqrt[3]{b}$, will give a rational Product.

Here we ſhall want only the firſt three Terms of the general Series, and here $n=3$. Therefore $\sqrt[n]{a^{n-1}} \mp \sqrt[n]{a^{n-2}b} + \sqrt[n]{a^{n-3}b} = \sqrt[3]{a^2} - \sqrt[3]{ab} + \sqrt[3]{b^2}$, is the Multiplier, but, for Conveniency, I ſhall make it the Multiplicand.—See the Operation.

$$\begin{array}{lrcrcrcr}
 & & & \sqrt[3]{a^2} & - & \sqrt[3]{ab} & + & \sqrt[3]{b^2} \\
 & & & & & \sqrt[3]{a} & + & \sqrt[3]{b} \\
\hline
 & & & + \sqrt[3]{a^2 b} & - & \sqrt[3]{ab^2} & + & b \\
 & a & - & \sqrt[3]{a^2 b} & + & \sqrt[3]{ab^2} & & \\
\hline
\text{The Product} & a & & * & & * & + & b. \\
\hline
\end{array}$$

EXAMPLE II.

Find the Surd, which multiplied by $\sqrt[4]{a^3}+\sqrt[4]{b^3}$ gives a rational Product.

Here $n=4$, and $\sqrt[n]{a^{3n-1}} \mp \sqrt[n]{a^{3n-2}b^3} + \sqrt[n]{a^{3n-3}b^6} \mp \sqrt[n]{a^{3n-4}b^9} = \sqrt[4]{a^9} - \sqrt[4]{a^6 b^3} + \sqrt[4]{a^3 b^6} - \sqrt[4]{b^9}$ is the Surd required.—See the Work.

$$\sqrt[4]{a^9} - \sqrt[4]{a^6 b} + \sqrt[4]{a^3 b^6} - \sqrt[4]{b^9}$$
$$\sqrt[4]{a^3} + \sqrt[4]{b^3}$$

$$+\sqrt[4]{a^9 b^3} - \sqrt[4]{a^6 b^6} + \sqrt[4]{a^3 b^9} - \sqrt[4]{b^{12}}$$
$$\sqrt[4]{a^{12}} - \sqrt[4]{a^9 b^3} + \sqrt[4]{a^6 b^6} - \sqrt[4]{a\,b^9}$$

The Product a^3 * * * $-\,b^3$.

Here all the Terms of the Product destroy one another, except $\sqrt[4]{a^{12}}$ and $-\sqrt[4]{b^{12}}$, and these Terms are respectively equal to a^3 and $-b^3$, therefore the Product is $a^3 - b^3$, rational.

EXAMPLE III.

Let it be required to find the Surd, which multiplied by $\sqrt{a} - \sqrt[3]{b}$, or $\sqrt[6]{a^3} - \sqrt[6]{b^2}$, shall give a rational Product.

Here $n=6$, and $\sqrt[n]{a^{3n-1}} + \sqrt[n]{a^{3n-2}b} + \sqrt[n]{a^{3n-3}b^4} + \sqrt[n]{a^{3n-4}b^6} + \sqrt[n]{a^{3n-5}b} + \sqrt[n]{a^{3n-6}b^{10}} = \sqrt[6]{a^{15}} + \sqrt[6]{a^{12}b^2} + \sqrt[6]{a^9 b^4} + \sqrt[6]{a^6 b^6} + \sqrt[6]{a^3 b^8} + \sqrt[6]{b^{10}}$, is the Surd required.—See the Multiplication.

$$\sqrt[6]{a^{15}} + \sqrt[6]{a^{12} b^2} + \sqrt[6]{a^9 b^4} + \sqrt[6]{a^6 b^6} + \sqrt[6]{a^3 b^8} + \sqrt[6]{b^{10}}$$
$$\sqrt[6]{a^3} - \sqrt[6]{b^2}$$

$$-\sqrt[6]{a^{15}b^2} - \sqrt[6]{a^{12}b^4} - \sqrt[6]{a^9 b^6} - \sqrt[6]{a^6 b^8} - \sqrt[6]{a^3 b^{10}} - \sqrt[6]{b^{12}}$$
$$\sqrt[6]{a^{18}} + \sqrt[6]{a^{15}b^2} + \sqrt[6]{a^{12}b^4} + \sqrt[6]{a^9 b^6} + \sqrt[6]{a^6 b^8} + \sqrt[6]{a^3 b^{10}}$$

$\sqrt[6]{a^{18}}$ * * * * * $-\sqrt[6]{b^{12}}$

Here the Product is $\sqrt[6]{a^{18}} - \sqrt[6]{b^{12}} = a^3 - b^2$, and is rational as required.

By this Reduction a Fraction which has a Binomial Surd for its Denominator, may be reduced to a more simple Form, by multiplying both its Numerator and Denominator by that Surd, which multiplied into its Denominator will give a rational Product.

Thus the Fraction $\dfrac{\sqrt{48}}{\sqrt{7}-\sqrt{3}}$ by multiplying its Numerator and Denominator by $\sqrt{7}+\sqrt{3}$, becomes $\dfrac{\sqrt{48}}{\sqrt{7}-\sqrt{3}} \times \dfrac{\sqrt{7}+\sqrt{3}}{\sqrt{7}+\sqrt{3}} = \dfrac{\sqrt{336}+\sqrt{144}}{7-3} = \dfrac{4\sqrt{21}+12}{4} = \sqrt{21}+3$.

SECTION XI.

Reduction of Equations which involve only one unknown Quantity.

AN Equation is a Proposition asserting the Equality of two Quantities differently expressed: Thus $x=ab$, is an Equation which shews that x is equal to the Product of a and b multiplied together.

Equations are the Means by which we come at such Conclusions as answer the Conditions of Problems.

CASE I.

Any Quantity may be transposed or moved from one Side of an Equation to the other, if its Sign be changed; this is nothing more in Effect than to add the same Quantity

tity with a contrary Sign on each side of the Equation; and it is evident, that if equal Quantities are added to equal Quantities their Sums must be equal.

Therefore transpose all the known Quantities to one Side of the Equation and all the unknown ones to the other Side, so shall the Value of the unknown Quantity be determined (where after Transposition its Co-efficient is Unity) as in the following

EXAMPLES.

Thus if $x-6=10$, then by transposing -6, with the contrary Sign, to the other side of Equation, we have $x=10+6=16$:

If $3x=2x+20$, then by transposing $2x$, we have $3x-2x=20$, or $x=20$.

If $5x+20=4x+30$, here by transposing 20 and $4x$, we have $5x-4x=30-20$, hence $x=10$.

If $7x-a+b=6x+c$; here by transposing $-a+b$ and $6x$, our Equation becomes $7x-6x=a-b+c$, hence $x=a-b+c$.

CASE II.

Any Quantity by which the unknown Quantity is multiplied may be taken away, if you divide every Term on both Sides of the Equation by it. For it is evident, that if equal Quantities are divided by equal Quantities, their Quotients must be equal.

Thus if $4x=24$; here dividing both sides of the Equation by 4, we get $x=6$, and the Equation $ax=ab$, divided in the same Manner by a, gives $x=b$.

If $ax=b$, then dividing both Sides by a, we get $x=\frac{b}{a}$.

If $5x+8=x+20$, then transposing 8 and x, by Rule I. we have $5x-x=20-8$; hence $4x=12$, and by Rule II. we get $x=\frac{12}{4}=3$.

Again, if $ax=b+c$, then $x=\frac{b+c}{a}$.

If

If $2ax+6ab=3c^2$, then tranſpoſing $+6ab$, we have $2ax=3c^2-6ab$, this Equation divided by $2a$, gives $x=\frac{3c^2}{2a}-3b$.

Sometimes the unknown Quantity is multiplied by different Co-efficients, ſome numeral and ſome literal, in which Caſe its Value may be found, by dividing both Sides of the Equation by the Sum of all the Co-efficients, connected together with their proper Signs.

Thus ſuppoſe the Equation after Tranſpoſition, &c. to be $ax+3x=b+c$, then dividing both Sides thereof by $a+3$ (the Sum of the Co-efficients of the unknown Quantity x) gives $x=\frac{b+c}{a+3}$.

In like Manner the Equation $ax+x=c+d$, being divided by $a+1$, gives $x=\frac{c+d}{a+1}$.

If $ax-x=c+d$, then $x=\frac{c+d}{a-1}$.

If $ax-2bx+cx=dr-n$, here dividing both Sides of this Equation by $a-2b+c$, gives $x=\frac{dr-n}{a-2b+c}$.

CASE III.

If the unknown Quantity is divided by any Quantity, that Quantity may be taken away if you multiply all the Terms of the Equation by it.

Thus if $\frac{x}{a}=b+2$, then multiplying every Term of the Equation by a, we have $\frac{ax}{a}=ab+2a$, or $x=ab+2a$.

If all the Terms of an Equation are divided by the ſame Quantity, let the common Diviſor be caſt away,

Thus

Thus if $\frac{x}{a} = \frac{b}{a}$, then by casting away the common Denominator a, the Equation $\frac{x}{a} = \frac{b}{a}$ becomes $x = b$.

In the same Manner the Equation $\frac{x}{c} = \frac{a}{c} - \frac{b}{c}$ will be reduced to $x = a - b$, and so of others.

CASE IV.

If the unknown Quantity be divided by different Divisors, let every Term of the Equation be multiplied by the Product of all those Divisors or Denominators, so shall the unknown Quantity be cleared of Fractions, as before.

Thus if $\frac{x}{c} + \frac{x}{a} = b$, then multiplying all the Terms of the Equation by ac, the Product of (a and c) the Denominators, we have $\frac{acx}{c} + \frac{acx}{a} = abc$, or $ax + cx = abc$; hence dividing by $a + c$, we get $x = \frac{abc}{a+c}$.

If $x + \frac{x}{2} = \frac{x}{3} + 7$; here by multiplying every Term by 2×3, or 6, the Product of the Denominators, we have $6x + \frac{6x}{2} = \frac{6x}{3} + 42$, or $6x + 3x = 2x + 42$, or $9x = 2x + 42$; hence by Rule I, $9x - 2x = 42$, or $7x = 42$, and by Rule II, we get $x = \frac{42}{7} = 6$.

If $\frac{x}{a} + \frac{5x}{b} - \frac{x}{c} = d$. This Equation being multiplied by abc, the Product of the Denominators, becomes

$bcx +$

$bcx+5acx-abx=abcd$, hence by Rule II. we get $x=\dfrac{abcd}{bc+5ac-ab}$.

C A S E V.

If the unknown Quantity be contained in a Surd Root, let all the other Terms be transposed to the contrary Side of the Equation, by Rule I. and then if both Sides be involved to the Power denominated by the Surd, an Equation will arise free from radical Quantities; unless there happen to be more than one Surd containing the unknown Quantity, in which Case the Operation must be repeated.

Thus, if $\sqrt{ax+b^2}=3b$; here, by squaring both Sides, we have $ax+b^2=9b^2$, and by transposing b^2, we have $ax=9b^2-b^2=8b^2$, hence, dividing by a, we get $x=\dfrac{8b^2}{a}$.

Let $\sqrt{x+a}=c-\sqrt{x+b}$; this Equation, by squaring each Side, becomes $x+a=c^2-2c\sqrt{x+b}+x+b$, hence by Transposition, we have $x-x+2c\sqrt{x+b}=c^2+b-a$, or $2c\sqrt{x+b}=c^2+b-a$, and dividing by $2c$ we get $\sqrt{x+b}=\dfrac{c}{2}+\dfrac{b-a}{2c}$, and by squaring again, we have $x+b=\overline{\dfrac{c}{2}+\dfrac{b-a}{2c}}\Big|^2$, therefore $x=\overline{\dfrac{c}{2}+\dfrac{b-a}{2c}}\Big|^2-b$.

Suppose $\sqrt[3]{ax-cx}=a$; here by cubing each Side, we have this Equation $ax-cx=a^3$, which, being divided by $a-c$, gives $x=\dfrac{a^3}{a-c}$.

C A S E VI.

If that Side of an Equation which contains the unknown Quantity be a compleat Square, Cube, or other Power, then will the Value of the unknown Quantity be obtained by extracting the proper Root from both Sides of the Equa-

tion; for if two Quantities are equal their Roots must necessarily be equal.

Suppose $x^2=25$, then will $x=\sqrt{25}=5$. For the Square Root of x^2 is x, and the Square Root of 25 is 5.

Let $x^2+2ax+a^2=b$; here, by extracting the Square Root on each Side, we have $x+a=\sqrt{b}$, and by transposing a, we get $x=\sqrt{b}-a$.

If $x^3=a+b$, then will $x=\sqrt[3]{a+b}$.

If $\overline{x+a}^n=c$; then by extracting the n^{th} Root, we have $x+a=\sqrt[n]{c}$, and therefore $x=\sqrt[n]{c}-a$.

CASE VII.

If the unknown Quantity be contained in the Terms of an Analogy, its Value may be found by multiplying Extremes and Means together for an Equation.

For if four Numbers are proportionable, the Product of the extreme Terms multiplied together, will be always equal to the Product of the two middle Terms so multiplied.

Thus suppose $x:16-x::3:5$. Then will the Product of the first and fourth Terms be $5x$, and the Product of ($16-x$ and 3) the two middle Terms, will be $\overline{16-x}\times3$, or $48-3x$, and by putting these two Products equal to each other, we have this Equation $5x=48-3x$, here by transposing $-3x$, we have $5x+3x=48$, or $8x=48$; hence $x=\frac{48}{8}=6$.

Again, if $x:a::b:c$; then by multiplying Extremes and Means, we have $cx=ab$, and consequently $x=\frac{ab}{c}$.

CASE VIII.

If the same Quantity be found on both Sides of an Equation, or multiplied into all the Terms, it may be struck out of the Equation. Thus, if $x+c=2a+c$; here the

the Quantity c, having the same Sign on both Sides of the Equation, I therefore reject it, and so $x=2a$, and it is evident that c would have been destroyed by Transposition.

Again, if $ax=3ab+ac$; here dividing every Term of the Equation by a, we have $x=3b+c$.

The Rules hitherto delivered relate to single Equations, which contain only one unknown Quantity, but when two, three, or more unknown Quantities are concerned in a Problem, then there must be as many Equations, which may be reduced to an Equation involving only one unknown Quantity, by the following Rules.

RULE I.

Observe which of all your unknown Quantities is the least involved, and let the Value of that Quantity be found in each Equation, by the Methods already explained, looking upon all the Rest as known; let the Values thus found be put equal to each other (for they are equal) because they all express the same Thing; whence new Equations will arise, out of which that Quantity will be wholly excluded; with these new Equations the Operation may be repeated; and the unknown Quantities exterminated one by one, till at last you come to an Equation containing only one unknown Quantity, which may be then solved by the preceding Rules.

Thus, let the given Equations be $x+y=9$, and $3x+5y=37$, to find x and y.

Here by transposing x in the first, and $3x$ in the second Equation, we have $y=9-x$, and $5y=37-3x$, hence $y=\frac{37-3x}{5}$: Now these two Quantities, namely $\frac{37-3x}{5}$ and $9-x$, are each found equal to y, and Quantities that are equal to one and the same Thing are evidently equal to one another, therefore $\frac{37-3x}{5}=9-x$; this Equation multiplied by 5, becomes $37-3x=45-5x$, hence by Transposition we get $2x=8$, therefore $x=\frac{8}{2}=4$; and $y=9-x=9-4=5$.

 RULE

RULE II.

Or let the Value of the unknown Quantity, which you would first exterminate, be found in that Equation wherein it is the least involved, considering all the other Quantities as known; and let this Value and its Powers, be substituted for that Quantity and its respective Powers in the other Equations; and with the new Equations thence arising repeat the Operation, till you have an Equation containing but one unknown Quantity.

Thus, let there be given the Equations $x+3y=9$, and $2\times x+7y=20$; then from the first Equation we shall have $x=9-3y$, and by writing $\underline{9-3y}$ for x in the second Equation there will arise $2\times\underline{9-3y}+7y=20$, or $18-6y+7y=20$; hence $y=20-18=2$, and therefore $x=9-3y=9-6=3$.

RULE III.

Or, lastly, let the given Equations be multiplied or divided by such Numbers or Quantities, whether known or unknown, that the Factor which involves the highest Power of the unknown Quantity to be exterminated, may be the same in each Equation, and then, by adding or subtracting the Equations, as Occasion may require, that Term shall be destroyed, and a new Equation will arise, wherein the Number of Dimensions, (if not the Number of unknown Quantities) will be diminished.

In solving the following Equations, I shall occasionally use the second and third of these three Rules, as these are generally more easy and expeditious than the first.

EXAMPLE I.

Given $\left\{\begin{matrix} x+y=s \\ x-y=d \end{matrix}\right\}$, to find x and y.

Add these two Equations together, and you will have $2x=d+s$, hence $x=\frac{d+s}{2}$; and by subtracting the second given Equation from the first, we get $2y=s-d$; therefore $y=\frac{s-d}{2}$.

EXAMPLE

EXAMPLE II.

Given $\left\{\begin{array}{l} x+3y=35 \\ 5x+y=105 \end{array}\right\}$

From five Times the firft Equation take the fecond, and you will have $14y=70$, hence $y=\frac{70}{14}=5$, and from three Times the fecond given Equation take the firft, and there will remain $14x=280$, whence $x=\frac{280}{14}=20$.

EXAMPLE III.

Given $\left\{\begin{array}{l} ax+by=c \\ dx+ey=f \end{array}\right\}$ to find x and y.

From e Times the firft Equation take b Times the fecond, and there will remain $aex-bdx=ce-bf$; hence $x=\frac{ce-bf}{ae-bd}$: Again from a Times the fecond Equation take d Times the firft, and you will have $aey-dby=af-dc$, whence $y=\frac{af-dc}{ae-db}$.

EXAMPLE IV.

Suppofe $\left\{\begin{array}{l} x+y=s \\ x:y::a:b \end{array}\right.$, hence $bx=ay$; this Equation being taken from b Times the firft, leaves $by=bs-ay$, hence $ay+by=bs$, therefore $y=\frac{bs}{a+b}$: Now write $\frac{bs}{a+b}$ inftead of y in the fecond Equation, namely in $bx=ay$, and you will have $bx=\frac{abs}{a+b}$, or $x=\frac{as}{a+b}$.

EXAMPLE V.

Suppofe $\left\{\begin{array}{r} 2x+y+2z=40 \\ 3x+3y-z=48 \\ x+z=2y \end{array}\right\}$

Take twice the third Equation from the first, and there will remain $y=40-4y$, or $5y=40$, therefore $y=\frac{40}{5}=8$; write 8 for y in the Sum of the second and third Equations, and there will arise $4x+3\times8=48+2\times8$, or $4x=40$; whence $x=\frac{40}{4}=10$; and by writing 10 for x and 8 for y in the third given Equation, you will have $10+z=2\times8$, hence $z=16-10=6$.

EXAMPLE VI.

$$\text{Suppose}\left\{\begin{array}{l} x+y=a \\ x+z=b \\ y+z=c \end{array}\right\}$$

From the Sum of the first and second Equations, take the third, and there will remain $2x=a+b-c$, therefore $x=\frac{a+b-c}{2}$: Again from the Sum of the first and third Equations take the second, and you will have $2y=a+c-b$, and $y=\frac{a+c-b}{2}$: Lastly, from the Sum of the second and third Equations subtract the first, and there will remain $2z=b+c-a$, hence $z=\frac{b+c-a}{2}$.

EXAMPLE VII.

$$\text{Given}\left\{\begin{array}{c} axy+3by^2+cyz^2=s \\ \frac{3y}{c}-\frac{z^2}{b}=d \\ x+\frac{2cz^2}{a}=e \end{array}\right\}\text{, to find } x, y \text{ and } z.$$

To the second Equation, multiplied by bcy, add the third Equation multiplied by ay, from the Sum, (namely $axy+3by^2+cyz^2=aey+bcdy$) subtract the first Equation, and you will have $aey+bcdy-s=0$, or $aey+bcdy=s$; and $y=\frac{s}{ae+bcd}$: The second Equation multiplied by bc, be-

comes

comes $3by-cz^2=bcd$, in this Equation write $\frac{s}{ae+bcd}$, for y, and you will have $\frac{3bs}{ae+bcd}-cz^2=bcd$, or $cz^2=\frac{3bs}{ae+bcd}-bcd$, hence $z^2=\frac{3bs}{aec+bc^2d}-bd$, and consequently $z=\sqrt{\frac{3bs}{aec+bc^2d}-bd}$; and by writing $\frac{3bs}{aec+bc^2d}-bd$, for z^2, in the third given Equation, there will arise $x+\frac{6bs}{a^2e+abcd}-\frac{2bcd}{a}=e$, whence $x=e+\frac{2bcd}{a}-\frac{6bs}{a^2e+abcd}$.

In this Solution the Signs of the Terms of the Equation $\frac{3bs}{ae+bcd}-cz^2=bcd$, were all changed, in the next Step, in order to bring z^2 out positive so that its Square Root might be subtracted.

The Signs of all the Terms of any Equation may be changed, and the Equality will still be retained; thus, if $-x=b-a$, then will $x=a-b$.

Observe always to bring out the unknown Quantities with the same Sign and Power in each Equation, when you equate their Values, as in the first of these three Rules, on Page 115.

SECTION XII.

OF ARITHMETICAL PROGRESSION.

ARITHMETICAL Proportion, or Progression, is when a Rank or a Series of Numbers or Quantities increase or decrease by a common Difference, or by a continual adding or subtracting some equal Number or Quantity.

As $\left\{ \begin{array}{l} 1, 3, 5, 7, 9, 11, 13. \\ 13, 11, 9, 7, 5, 3, 1. \end{array} \right\}$ Here the common Difference is 2.

CASE I.

If the first Term of a Series of Quantities be a, and if the common Difference by which each succeeding Term increases, or decreases, be denoted by d, then will the second Term be $a+d$, the third $a+2d$, the fourth $a+3d$, &c. in an increasing Series; if x be the first Term of a decreasing Series, then will the second Term be $x-d$, the third $x-2d$, the fourth $x-3d$, &c. and these Serieses expressed in a general Manner will stand thus:

$\left\{ \begin{array}{l} \text{Increasing, } a, a+d, a+2d, a+3d, a+4d, a+5d, \text{\&c.} \\ \text{Decreasing, } x, x-d, x-2d, x-3d, x-4d, x-5d, \text{\&c.} \end{array} \right.$

The Sum of the first Series continued to six Terms, is $6a+15d=3\times\overline{2a+5d}$, and the Sum of the first six Terms of the second Series is $6x-15d=3\times\overline{2x-5d}$.

Add the first Term a, in the first Series, to the sixth Term $a+5d$, and you will have $2a+5d$, for the Sum of the Extremes of six Terms of that Series; and the first Term x in the second Series added to $x-5d$ the sixth Term thereof, gives $2x-5d$; so that the Sum of the Extremes in each Series being multiplied by 3, half the Number of Terms, gives $3\times\overline{2a+5d}$ and $3\times\overline{2x-5d}$, the Sum of all the Terms in each Series respectively, the very same as before.

And since this will ever be the Case, let the Number of Terms be what it will, it is plain that the Sum of any Series of Terms in Arithmetical Progression is equal to the Sum of the first and last Terms multiplied by half the Number of Terms.

Put $s=$ the Sum of all the Series, and $n=$ the Number of Terms, then since d does not commence till the second Term, it is evident that the Co-efficient of d in the last Term will be $n-1$; therefore the last Term will be $a+\overline{n-1}\times d$, or $a+nd-d$, which added to the first Term a, gives

a, gives $2a+nd-d$, for the Sum of the Extremes, which being multiplied by $\frac{n}{2}$, gives $\frac{2an+n^2d-nd}{2}$, or $\overline{a+\frac{nd-d}{2}}\times n$, for the Sum of the whole Series.

Hence $s=\overline{a+\frac{nd-d}{2}}\times n$. To give an Inſtance of this general Theorem, let it be required to find the Sum of the Series $3+6+9+12+15$, &c. continued to an Hundred and Fifty Terms.

Here $a=3$, $d=3$, $n=150$; and $s=\overline{a+\frac{nd-d}{2}}\times n=\overline{3+\frac{450-3}{2}}\times 150=33975$.

N. B. The Sum of this Series is the Anſwer to the third Queſtion in Mr. G. Dyer's Arithmetical Progreſſion, in his School's Aſſiſtant, Page 86, firſt Edition, where his Anſwer is 22725 Yards, or 12 Miles, 7 Furlongs, 65 Yards; inſtead of 33975 Yards, or 19 Miles, 2 Furlongs, and 95 Yards, as it ought to have been.

CASE II.

If the firſt Term of the Series be (0), a Cypher, or nothing, then ſhall the Sum of the Series be equal to the laſt Term $a+nd-d$, multiplied by $\frac{n}{2}$, half the Number of Terms.

For Example, the Sum of the Series $0+1+2+3+5+6+7+8+9=9\times\frac{10}{2}=\overline{a+nd-d}\times\frac{n}{2}$.

For here $a=0$, $d=1$, and $n=10$; therefore in this Caſe we have $s=\overline{nd-d}\times\frac{n}{2}=9\times\frac{10}{2}=9\times 5=45$.

CASE

CASE III.

Any three Quantities in Arithmetical Progression, the double of the middle Quantity, is equal to the Sum of the Extremes.

Thus in the three Terms a, $a+d$, $a+2d$, it is evident by Inspection, that the Sum of (a and $a+2d$) the Extremes, is ($2a+2d$) equal to twice $a+d$ the mean or middle Term.

SECTION XIII.

Of Geometrical Proportion or Progression.

WHEN a Series of Quantities increase by a constant Multiplier, or decrease by a constant Divisor, they are said to be in Geometrical Proportion; and the said Multiplier or Divisor is called the Ratio of the Series.

CASE I.

As 1. 2. 4. 8. 16. 32. 64. Here the common Multiplier, or Ratio, is 2.

Also 729. 243. 81. 27. 9. 3. 1. Here the common Divisor, or Ratio, is 3.

And, in general, if the first Term of the Series be denoted by a, and the Ratio by r, then will the Series, if it be an increasing one, st and thus, $a+ar+ar^2+ar^3+ar^4+ar^5$, &c. If it be decreasing, thus $a+\frac{a}{r}+\frac{a}{r^2}+\frac{a}{r^3}+\frac{a}{r^4}+\frac{a}{r^5}$, &c.

CASE II.

In any Series of Geometrical Proportionals, the Product of the first and last Terms is equal to the Product of any two others, equally distant from them; and the Square of

of the middle Term if the Number of Terms be odd, is equal to the Product of the Extremes.

Thus, if we take six Terms we shall have $a \times ar^5 = ar \times ar^4 = ar^2 \times ar^3 = a^2 r^5$.

Or, if we take four Terms, we shall have $a \times ar^3 = ar \times ar^2 = a^2 r^3$. If we take three Terms, then will $a \times ar^2 = ar \times ar$. If we take five Terms, then will $a \times ar^4 = ar^2 \times ar^2 = ar \times ar^3 = a^2 r^4$; and so of others.

CASE III.

Put $s =$ the Sum of all the Terms, and $n =$ the Number of Terms; then since the Exponent of r begins from the second Term, it is evident that the Exponent of r in the last Term will be $n-1$, and therefore the last Term will be ar^{n-1} (in any increasing Geometrical Series) the last Term but one will be ar^{n-2}, the last but two ar^{n-3}, &c.

Hence we have $s = a + ar + ar^2 + ar^3 \ldots + ar^{n-3} + ar^{n-2} + ar^{n-1}$, and $s = a + ar + ar^2 + ar^3 + ar^4 \ldots + ar^{n-2} + ar^{n-1}$.

Here the fifth Terms (namely ar^4 and ar^{n-3}) in each Equation are equal to one another, though differently expressed, and by taking the second Equation from r Times the first, you will have $rs - s = ar^n - a$, whence $s = \dfrac{ar^n - a}{r-1}$.

Or by making the Equations different only in the Expressions of their last Terms, (which may always be done, let the Number of Terms be what it will) we have $s = a + ar + ar^2 + ar^{n-1}$, and $s = a + ar + ar^2 + ar^3$: From r Times the first of these two Equations take the second, and you will have $rs - s = ar^n - a$, and $s = \dfrac{ar^n - a}{r-1}$, as before.—See the Solution to the 85 promiscuous Question; but this general Theorem may be differently expressed:

For $s = \dfrac{ar^n - a}{r-1} = \dfrac{ar^{n-1} - a}{r-1} + ar^{n-1}$.

And this Rule is frequently used by Arithmeticians in finding the Sum of all the Terms in a Series of Numbers in Geometrical Progression; which in Words is thus: From the last Term take the first, divide the Remainder

by

by the Ratio less one, and to that Quotient add the last Term, gives the Sum required.—See my 84 promiscuous Questions in Arithmetic.

CASE IV.

The Square of the greater Mean of a, ar, ar^2, ar^3, four Numbers in Geom. Prog, divided by the less Mean, gives the greater Extreme, and the Square of the less Mean divided by the greater, gives the less Extreme, thus $\frac{a^2r^4}{ar}=ar^3$, and $\frac{a^2r^2}{ar^2}=a$.

CASE V.

The Ratio of two homogeneous Quantities may be determined without the Intervention of a third, by dividing the Antecedent by the Consequent.

Thus the Ratio of 6 to 3 is $\frac{6}{3}=\frac{2}{1}=2$, or double, the Ratio of a to b is $\frac{a}{b}$, and the Ratio of ar to r is a.

CASE VI.

To find the Sum of a decreasing Series of Geometricals Proportionals.

Let $s=a+\frac{a}{r}+\frac{a}{r^2}+\frac{a}{r^3}$, and $s=a+\frac{a}{r}+\frac{a}{r^2}+\frac{a}{r^{n-1}}$.

This Equation subtracted from r Times the first, leaves $rs-s=ar-\frac{a}{r^{n-1}}$, whence we have $s=\frac{ar-\frac{a}{r^{n-1}}}{r-1}$ for the Sum of a finite decreasing Series; to illustrate this, let n (or the Number of Terms) $=6$, $a=729$, and $r=3$, then we

we shall have $s = \dfrac{ar - \dfrac{a}{r^{n-1}}}{r-1} = \dfrac{2187 - \dfrac{729}{243}}{3-1} = \dfrac{2184}{2}$ $= 1092.$

CASE VII.

But if the Number of Terms in such a Series is supposed to be infinite, then since the Value of each succeeding Term perpetually decreases, it is evident that the last Term $\frac{a}{r^{n-1}} = 0$, for a finite Qnantity (a) divided by (r^{n-1}) an infinite one is nothing; and therefore we have $s = \dfrac{ar}{r-1}$ for the Sum of an infinite decreasing Series.

To exemplify this, let it be proposed to find how far a Body would move at this Rate, namely, in the first Moment 10 Miles, in the second 9 Miles, in the third $8\frac{1}{10}$, and so on eternally, as 10 to 9.

Here is given $a = 10$, and $r = \frac{10}{9}$; whence we have $s = \dfrac{ar}{r-1} = \dfrac{10 \times \frac{10}{9}}{\frac{10}{9} - 1} = \dfrac{100 \div 9}{1 \div 9} = 100$ Miles.

So that if a moving Body were to continue its Motion eternally, in the given Ratio, it would only run out 100 Miles, or more than any Thing that is less than 100 Miles.

CASE VIII.

The Sum of the Extremes of a, ar, ar^2, ar^3, four Numbers in Geom. Prog. multiplied by their Product, is equal to the Sum of the Cubes of the two Means, thus $\overline{a + ar^3} \times a \times ar^3 = a^3r^3 + a^3r^6$.

CASE IX.

The Square of the second Term divided by the first, gives the third Term, thus $\dfrac{a^2 r^2}{a} = ar^2$.

SECTION

SECTION XIV.

Of Harmonic, or Musical Proportion.

CASE I.

THREE Terms are said to be harmonically proportional, when the Difference of the first and second is to the Difference of the second and third, as the first is to the third.

Thus 2, 3, 6, are harmonically proportional: For $1 : 3 :: 2 : 6$; so that if two Terms of an Harmonic Proportion be given, the third may be readily found: Thus if a, b, x, be harmonically proportional, then it will be $a-b : b-x :: a : x$, hence by multiplying Extremes and Means, we get $ax-bx=ab-ax$; hence we have $2ax-bx=ab$, and $x=\frac{ab}{2a-b}$.

CASE II.

When four Terms are harmonically proportional, the Difference of the first and second is to the Difference of the third and fourth, as the first is to the fourth.

Thus 10, 16, 24, 60, are Harmonic Proportionals: For here the Difference between the first and second Term is 6, the Difference between the third and fourth is 36, and as $6 : 36 :: 10 : 60$.

Whence if three Terms of such Harmonic Proportionals, are given, a fourth is easily found.

For if a, b, c, x, are Harmonic Proportionals, then $a-b : c-x :: a : x$; hence $ax-bx=ac-ax$, and $2ax-bx=ac$; therefore $x=\frac{ac}{2a-b}$.

SECTION

SECTION XV.

Of Algebraic Problems, or the Solution of Questions producing simple Equations.

SIMPLE Equations are those wherein the unknown Quantity is only of one Dimension.

When a Problem is proposed to be solved algebraically, its due Signification ought, in the first place, to be perfectly understood, and if the Number of Quantities, given from the Conditions of the Question, be equal to the Number of Quantities sought, then the Question is truly limited; but if the Number of Quantities required exceeds those given, then the Question frequently admits of many Answers, and is therefore called unlimited.

DIRECTION.

In the Solution of Problems, having put Letters for the Numbers required, you must add to, or subtract from them any Quantity as the Question directs, and also multiply or divide them by any Number, and make the Sum, Difference, Product, or Quotient, &c. in each Case equal to their respective given Numbers; and then solve the Equations by the preceding Rules.

But since Examples improve more than any Precepts that can be given, I shall therefore proceed to

QUESTION I.

Wherein let it be required to find a Number, which being multiplied by 3, and having 7 added to the Product, the Sum shall be 100.

Put x for the Number sought, then multiplying x by 3, and adding 7 to the Product $3x$, the Sum is $3x+7$, which, by the Question, is equal to 100, whence we have this Equation $3x+7=100$, hence we get $3x=100-7=93$, and $x=\frac{93}{3}=31$.

QUESTION

QUESTION II.

What Number is that, which being divided by 4, and having 9 ſubtracted from the Quotient, the Remainder ſhall be 20.

Put x for the Number required, then dividing x by 4, and ſubtracting 9 from the Quotient $\frac{x}{4}$, the Remainder is $\frac{x}{4}-9$, and is, by the Queſtion, equal to 20, whence $\frac{x}{4}-9=20$, hence $\frac{x}{4}=29$, and multiplying by 4, we get $x=116$.

QUESTION III.

A Gentleman diſtributing ſome Money among Children on Valentine's Day, wanted Eight-pence to give them Three-pence a-piece; he therefore gave to each Two-pence, and had Sixpence remaining: How many Children were there?

Put x for the Number of Children, then the Gentleman wanted Eight-pence to give $3x$ Pence among them; he had therefore $3x-8$ Pence, at firſt, out of theſe he gave $2x$ Pence, and the remaining Pence $3x-8-2x$, or $x-8$ are equal to 6, by the Queſtion, hence we have $x-8=6$, and $x=14$, the Number of Children required.

QUESTION IV.

To find two Numbers whoſe Sum ſhall be equal to the Difference of their Squares, and five Times the greater added to three Times the leſs, ſhall make 45.

Put x for the greater and y for leſs Number; then per Queſtion we have $x^2-y^2=x+y$, and $5x+3y=45$: Divide the firſt Equation by $x+y$, and you will have $x-y=1$, hence $x=y+1$, now write $y+1$, for x in the ſecond Equation, and there will ariſe $5y+5+3y=45$, whence $8y=40$, and $y=\frac{40}{8}=5$, therefore $x=y+1=5+1=6$.

By

By this Solution it appears that when the Sum of two Numbers is equal to the Difference of their Squares, that the Difference of the Numbers themselves is equal to 1.

QUESTION V.

Two Persons, A and B were talking of their Ages: Says A to B, seven Years ago I was thrice as old as you at that Time; and seven Years hence I shall be just twice as old as you will be: I demand their present Ages?

Let the present Age of A be denoted by x, and that of B by y; then their Ages 7 Years ago were $x-7$ and $y-7$, and 7 Years hence they will be $x+7$ and $y+7$; hence per Question we have $x-7=\overline{y-7}\times 3$, and $x+7=\overline{y+7}\times 2$: From the first Equation we get $x=3y-14$, and from the second $x=2y+7$, therefore $3y-14=2y+7$; hence $y=21$, and $x=2y+7=42+7=49$.

QUESTION VI.

An unknown Sum of Money is to be divided equally among an unknown Number of Men: Now if there were three Men less, each Man would have 150l. more; but if there were six Men more, then each Man would have 120l. less. I demand the Sum of Money and the Number of Men?

Put x for the Money that each Man must receive, and y for the number of Men sought, then xy is the Sum required; which, being divided by the Number of Men less 3, the Quotient $\frac{xy}{y-3}$, must by the Nature of the Question be equal to $x+150$, each Man's Share more 150, hence we have $\frac{xy}{y-3}=x+150$, and, by the same way of Reasoning, $\frac{xy}{y+6}=x-120$; here multiplying the first Equation by $y-3$, we have $xy=xy+150y-3x-450$, hence $x=50y-150$, and multiplying the second Equation by $y+6$, we get $xy=xy-120y+6x-720$, whence $x=20y+120$, therefore $50y-150=20y+120$, hence we get $y=9$,

 the

the Number of Men, therefore $x=20y+120=180+120$ $=300$l. and $xy=300\times 9=2700$l. the Sum to be divided.

QUESTION VII.

Required that vulgar Fraction whose Numerator lessened by 1, shall be $\frac{2}{3}$; but the Denominator being lessened by 1, shall make the said Fraction $\frac{5}{7}$.

Let $\frac{x}{y}$ be the Fraction sought, then per Question $\frac{x-1}{y}=\frac{2}{3}$ and $\frac{x}{y-1}=\frac{5}{7}$; Multiplying the first Equation by $3y$ and the second by $\overline{y-1}\times 7$, we have $3x-3=2y$ and $7x=5y-5$, whence $y=\frac{3x-3}{2}$ and $y=\frac{7x+5}{5}$, therefore $\frac{3x-3}{2}=\frac{7x+5}{5}$, hence we get $x=25$, and $y=\frac{3x-3}{2}=\frac{75-3}{2}=36$, and therefore $\frac{x}{y}=\frac{25}{36}$, the Fraction required.

Or thus, from the first Equation multiplied by $15y$, take the second multiplied by $\overline{y-1}\times 14$, and you will have $x-15=10$, hence $x=25$; write 25 instead of x in the first Equation, and there will arise $\frac{24}{y}=\frac{2}{3}$, whence $2y=72$, and $y=36$ as before.

QUESTION VIII.

There is a certain Number consisting of 2 Figures, and it is equal to four Times the Sum of its Digits (namely the Sum of the two Figures) and if you add eighteen to the Number, the Digits will be inverted, that is, the first Figure will stand last, and the last first: I demand the Number.

Put x for the Figure in the ten's Place, and y for that in the Place of Units, then will $10x+y$ be the Number sought, whence per Question $10x+y=\overline{x+y}\times 4$; and in order to invert the Figure y from the Units to the ten's Place,

Place, multiply it by 10, and you will, by the Question, have $10y+x=10x+y+18$: From the first Equation we get $y=2x$, and from the second $y=x+2$; therefore $2x=x+2$, hence $x=2$, and $y=2x=4$; and therefore $10x+y=20+4=24$, the Number required.

QUESTION IX.

A Lady being asked how old she was, replied thus: My Age, if multiplied by three, and if two Sevenths of that Product be tripled, the Square Root of two Ninths of this will be four: Quere the Lady's Age?

Put $7x^2$ for the Lady's Age in Years, then three Times her Age will be $21x^2$, and $\frac{2}{7}$ of $21x^2$, the Product is $6x^2$, this tripled becomes $18x^2$, and $\frac{2}{9}$ of $18x^2$ is $4x^2$, whose square Root $2x$, is equal to 4 by the Question, hence $2x=4$, therefore $x=2$, and consequently $x^2=4$, therefore $7x^2=28$ Years, the Age required.

QUESTION X.

Divide 100 twice in two Parts, so that the major Part of the first Division may be three Times the minor Part of the second Division, and the major Part of the second may be double the minor Part of the First.

Put $3x$ for the major Part of the first Division, then will $100-3x$ be the minor Part; and x will be the minor Part of the second Division, therefore $100-x$ is the major Part, which is, by the Question, equal to twice $100-3x$, the minor Part of the first Division, whence $100-x=200-6x$, hence we get $x=\frac{100}{5}=20$, the minor Part of the second Division, and $100-20=80$ is the major Part: Again, $3x=3\times20=60$, is the major of the first Division, and therefore $100-60=40$ is the minor Part.

QUESTION XI.

A Greyhound seeing a Hare at the Distance of 50 of his own Leaps, pursues her full Speed, making 3 for every 4 of the Hare's, and passing over as much Ground in 2

 Leaps

Leaps as the Hare did in 3: I demand how many Leaps each made during the whole Course.

Put $3x$ for the Dog's Leaps, then will the Hare's be $4x$; and $3x-50 : 4x :: 2 : 3$, hence $9x-150=8x$, and $x=150$, therefore $3x=450$ Leaps of the Dog's, and $4x=600$ Leaps, that the Hare made.

QUESTION XII.

Divide 144 into four Parts, in such a Manner that if to the first you add 5, from the second subtract 5, multiply the third by 5, and divide the fourth by 5, the Sum, Differenee, Product, and Quotient shall be all equal.

Put x for the first Part, then it is evident that $x+10$ will be the second, $\frac{x}{5}+1$, the third, and $5x+25$, the fourth Part: The Sum of these Parts is $7x+\frac{x}{5}+36$, and is equal to 144 by the Question, hence $7x+\frac{x}{5}+36=144$, this Equation solved, gives $x=15$; the other three Parts are 25, 4 and 100.

Or more generally thus, put $a=5$, and $s=144$, then will $x+x+2a+\frac{x}{a}+1+ax+a^2=s$, hence multiplying by a, we get $a^2x+2ax+x+a^3+2a^2+a=as$, therefore $a^2x+2ax+x=as-a^3-2a^2-a$, and $x=\frac{as-a^3-2a^2-a}{a^2+2a+1}=\frac{720-125-50-5}{25+10+1}=\frac{540}{36}=15$: By writing $\frac{as-a^3-2a^2-a}{a^2+2a+1}$ for x, in the second Part $(x+2a)$ it becomes $\frac{as-a^3-2a^2-a}{a^2+2a+1}+2a$, or $\frac{as+a^3+2a^2+a}{a^2+2a+1}=\frac{900}{36}=25$.

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In like Manner the third Part $\frac{x}{a}+1$, becomes $\frac{s-a^2-2a-1}{a^2+2a+1}+1$, or $\frac{s}{a^2+2a+1}=\frac{144}{36}=4$; and the fourth Part $ax+a^2$ will be transformed to $\frac{a^2s-a^4-2a^3-a^2}{a^2+2a+1}+a^2$, or $\frac{a^2s}{a^2+2a+1}=\frac{3600}{36}=100$.

QUESTION XIII.

Find three Numbers, so that the first and half the Remainder, the Second and one third of the Remainder, the Third and one fourth of the Remainder, may always make 85.

Put $a=85$, and let x, y and z denote the three required Numbers; then per Question,

$$x+\frac{y+z}{2}=a,$$

$$y+\frac{x+z}{3}=a,$$

$$z+\frac{x+y}{4}=a;$$

From three Times the second Equation take twice the first, and you will have $2y-x=a$, or $x=2y-a$; again, from six Times the second Equation take twice the first, and there will remain $5y+z=4a$, whence $z=4a-5y$: Now write $2y-a$ for x, and $4a-5y$ for z, in four Times the third Equation, and there will arise $15a-17y=4a$, hence $y=\frac{11a}{17}\left(=\frac{935}{17}\right)=55$, therefore $x\ (=2y-a)=\frac{22a}{17}-a=\frac{5a}{17}=25$; and $z\ (=4a-5y)=4a-\frac{55a}{17}=\frac{13a}{17}=65$.

QUESTION XIV.

A General ranging his Army in a square Battalia, finds he has 284 Soldiers to spare; but increasing the Side of the Square with one Man, he wants 25 to fill up the Square: How many Soldiers had he?

Put x for the Men in rank and file, then will the Square of these be x^2, and having 284 Men to spare, he had x^2+284; but increasing rank and file by 1 Man, the Side is $x+1$, whose Square x^2+2x+1 less 25, must by the Question be equal to x^2+284, hence $x^2+2x+1-25=x^2+284$, or $2x=308$; therefore $x=\frac{308}{2}=154$, and $x^2+284=24000$, the Number of Men sought.

QUESTION XV.

To find that Number, which being divided into four equal Parts, the Product of those Parts shall be equal to the Product of five equal Parts of the same Number multiplied together.

Put x for the required Number, then will $\frac{x}{4}$ be one fourth and $\frac{x}{5}$ one fifth Part thereof, hence per Question we have $\frac{x}{5}\times\frac{x}{5}\times\frac{x}{5}\times\frac{x}{5}\times\frac{x}{5}=\frac{x}{4}\times\frac{x}{4}\times\frac{x}{4}\times\frac{x}{4}$, or $\frac{x^5}{3125}=\frac{x^4}{256}$, therefore $x^5=\frac{3125x^4}{256}$; and $x=\frac{3125}{256}$ $=12\frac{53}{256}$.

QUESTION XVI.

A Waterman rowed in the River Orwell from Ipswich to Harwich and back again (without Intermission) in five Hours; and in coming from Harwich, meeting an equal Tide, with an uniform stroke, finds he can row 8 Miles with the Tide to 3 Miles against it; Now if the Distance between

between Ipswich and Harwich be 10 Miles, at what Rate an Hour did the Tide move?

Put $a=10$, $b=3$, $c=8$, $t=5$, $v=$ the Velocity of the Tide per Hour in Miles, and $x=$ the whole Time in which he rowed with the Tide, then will $t-x$ be the Time he rowed against it, and $x : a :: 1 : \frac{a}{x}$ his Velocity per Hour with the Tide, and $t-x : a :: 1 : \frac{a}{t-x}$ his Celerity per Hour against it, and by the Solution to the 30 Miscellaneous Questions, we shall have $v=\frac{a}{2x}-\frac{a}{2t-2x}$; and because his Velocity with the Tide was to that against it as 8 to 3, it is evident his Time of rowing with the Tide was to that against it as 3 to 8, therefore it will be $x : t-x :: b : c$, hence we get $cx=bt-bx$, and $x=\frac{bt}{b+c}$, and by writing $\frac{bt}{b+c}$, for x, in the Equation $v=\frac{a}{2x}-\frac{a}{2t-2x}$, we have $v\left(=\frac{a}{\frac{2bt}{b+c}}-\frac{a}{2t-\frac{2bt}{b+c}}\right)$ $=\frac{ab+ac}{2bt}-\frac{ab+ac}{2ct}=2\frac{7}{24}$ Miles per Hour, the required Velocity of the Tide.

QUESTION XVII.

A Gentleman having two Daughters, gave to one of them a round Piece of Land, and to the other a square Piece; these Lands were each valued at fifteen Pounds per Acre; and it was found by Mensuration, that the Inches which compassed each Piece were equal in Number to the Shillings that paid for it: Quere, which of these Ladies had the best Fortune?

First, 15l. : 6272640 (the square Inches in an Acre) :: 1s. : 20908,8, the square Inches that one Shilling will

 pay

pay for; and because the Inches that compass each Piece are respectively equal in Number to the Shillings that paid for it, therefore the Perimeter of the Square and the Circumference of the Circle are each to their respective Areas as 1 to 20908,8: Put $a=20908,8$, $c=3,14159265$, and $2x=$ the Diameter of the Circle, in Inches, then will $2cx$ be its Circumference, and its Area will be cx^2 (for Half the Circumference multiplied by half the Diameter, gives the Area) and by what has been premised, it will be $2cx : cx^2 :: 1 : a$, hence $cx^2 = 2acx$, or $cx=2ac$, and consequently $2cx=4ac=262747,7296$ + Shillings, or 13137l. 7s. 8¾d. the Lady's Fortune that had the circular Estate: Put y for the Side of the Square, then will $4y$ be its Perimeter and y^2 will be its Area; and $4y : y^2 :: 1 : a$, hence we get $y^2=4ay$, therefore $4y=16a=334540,8$ Shillings, or 16727l. 0s. 9½d. the Fortune of the amiable Fair that had the Square, which exceeds the other Lady's Fortune by 3589l. 13s. 0¾d.

QUESTION XVIII.

In what Time will the simple Interest of 400l. become equal to the Discount of 520l. both being calculated at 5 per Cent. per Annum?

Put $a=400$, $b=520$, $c=100$, $r=5$, the rate per Cent, and $x=$ the Time required in Years: Then $c : r :: a : \frac{ar}{c}$, the Interest of 400l. for one Year, and $\frac{arx}{c}$ is its Interest for the Time x; and by the Rule of Discount it will be $c+rx : rx :: b : \frac{brx}{c+rx}$ the Discount of 520l. for the same Time x; which must by the Question be equal to the Interest of 400l. for x Years; whence $\frac{arx}{c} = \frac{brx}{c+rx}$, or $\frac{a}{c} = \frac{b}{c+rx}$, this Equation multiplied by $c \times \overline{c+rx}$, becomes $arx+ac=bc$, hence $x=\frac{bc-ac}{ar}=6$ Years.

QUESTION

QUESTION XIX.

If 18 Oxen in 5 Weeks will eat 6 Acres of Graſs, and 45 Oxen will eat 21 Acres in 9 Weeks, how many Oxen will eat 38 Acres in 19 Weeks, the Graſs being allowed to grow uniformly?

Let Unity denote the Graſs eaten by an Ox in a Week: Put $a=18$, $b=5$, $c=6$, $d=45$, $m=21$, $n=9$, $r=38$, $s=19$, $x=$ the Number of Oxen ſought, $w=$ the Graſs on an Acre at firſt, $z=$ the Increaſe of the Graſs upon an Acre per Week (after the firſt 5 Weeks), $p=4$, the Exceſs of 9 Weeks above 5, and $t=14$, the Difference between 19 Weeks and 5: Then will rtz, be the Increaſe on 38 Acres in 14 Weeks, and rw is the Graſs upon 38 Acres at firſt; the Sum of theſe muſt by the Queſtion be equal to what x Oxen eat in 19 Weeks, hence $sx=rw+rtz$: Again, mpz is the Increaſe on 21 Acres in 4 Weeks, and mw is the graſs upon 21 Acres at firſt, theſe together are equal to dn; for if an Ox eats a Quantity of Graſs in a Week, denoted by Unity, then d Oxen in n Weeks will eat dn Times as much, therefore $mw+mpz=dn$, and cw, the Graſs upon ſix Acres at firſt, is equal to ab, what 18 Oxen eat in 5 Weeks, whence $cw=ab$; therefore $w=\frac{ab}{c}$; to the foregoing Equation $sx=rw+rtz$, multiplied by mp, add the Equation $mw+mpz=dn$, multiplied by rt, and you will have $mpsx+mrtw+mprtz=dnrt+mprw+mprtz$, or $mpsx=dnrt+mprw-mrtw$; write $\frac{ab}{c}$ for w, and you will have $mpsx=dnrt+\frac{abmpr}{c}-\frac{abmrt}{c}$, or $cmpsx=cdnrt+abmpr-abmrt$, and conſequently $x=\frac{cdnrt+abmpr-abmrt}{cmps}\left(=\frac{574560}{9576}\right)=60$.

By this Theorem, my 69th Miſcellaneous Queſtion in the Arithmetic may be readily ſolved

Here follow Queſtions, in the Solution of which the Values of the unknown Quantities are found by extracting their proper Roots.

QUESTION

QUESTION XX.

Find two Numbers, such that the Square of their Product, added to their fourth Powers, shall make 34048, and the Sum of their Squares, added to their Product shall be 304.

Put x and y for the two required Numbers, then per Question we have $x^4+x^2y^2+y^4=34048$, and $x^2+xy+y^2=304$; dividing the first Equation by the second, we get $x^2-xy+y^2=112$, which, taken from the second Equation, leaves $2xy=192$, or $xy=96$, this being added to the second Equation, and subtracted from the third, we have $x^2+2xy+y^2=400$, and $x^2-2xy+y^2=16$; whence by extracting the Square Root, we get $x+y=\sqrt{400}=20$, and $x-y=\sqrt{16}=4$; this Equation added to, and subtracted from $x+y=20$, gives $2x=24$, or $x=12$, and $y=8$.

QUESTION XXI.

Sixty Thousand Soldiers were placed two Yards and three Quarters asunder on a Plain, in the Form of a long Square, whose Breadth to its Length is as 2 to 3. How many Men were there in rank and in file, and on how many Acres of Land did they stand?

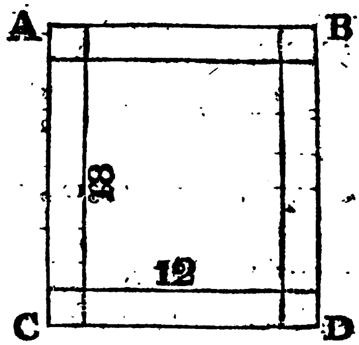
Put $3x$ for the Length, then will $2x$ be the Breadth, and by the Question we have $3x\times2x=60000$, or $6x^2=60000$, therefore $x^2=10000$, hence $x=\sqrt{10000}=100$, therefore $3x=300$ Men in rank, and $2x=200$ Men in file:

Now it is evident that the Intervals or Spaces among the Men in rank and file are less by one than the Number of Men in each, therefore $\overline{300-1}\times2{,}75=822{,}25$ Yards the Length, and $\overline{200-1}\times2{,}75=547{,}25$ Yards the Breadth; then $822{,}25\times547{,}25=449976{,}3125$ the Area in Square Yards, which being divided by 4840, the Square Yards in an Acre, gives $92{,}970312$, or 92 Acres, 3 Roods and $35{,}24992$ Perches, for the Ground which they stood upon.

QUESTION

QUESTION XXII.

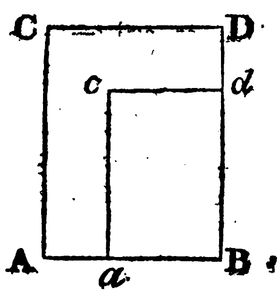
Suppose the Plate of a Looking-glass is 18 Inches by 12, and is to be framed with a Frame of equal Width, whose Area is to be equal to that of the Glass, the Width of the Frame is required?



Put $x+3=AC$ the Length of the Glass and Frame, then will $x-3=AB$, their Breadth; now the Area of the Frame and the Glass together, must by the Question be equal to twice 18×12, the Area of the Glass alone, whence we have $\overline{x+3}\times\overline{x-3}=18\times12\times2$, or $x^2-9=432$, hence $x^2=441$, therefore $x=\sqrt{441}=21$; and $x+3=24$ Inches, the Length of the Glass and twice the Breadth of the Frame, therefore the Breadth of the Frame is $\left(\frac{24-18}{2}, \text{ or}\right)$ 3 Inches.

QUESTION XXIII.

A Gentleman has a Garden 100 Feet long, and 80 broad in which he would make a Walk of an equal Width half round (as per Figure) the Area of the Walk is to be equal to half that of the whole Garden; Quere the Breadth of the Walk?



Since the Length and Breadth of the Garden are equally diminished by the Walk (A a C c D d) therefore A C$-$C D$=ac-cd$ ($=20$); put $n=10$, half the Difference of the Sides, A B$=80=r$, A C$=100=s$, $x+n=ac$, then will $x-n=a$ B, and (A B$-a$ B $=$) A $a=r+n-x$, and by the Question, the Area of the Parallelogram $a\,c\,d$ B, must be equal to half ($r\,s$) the Area of the whole Garden A B C D, whence we have $\overline{x+n} \times \overline{x-n} = \frac{rs}{2}$, or $x^2 - n^2 = \frac{rs}{2}$, hence $x^2 = \frac{rs}{2} + n^2$, therefore $x = \sqrt{\frac{rs}{2} + n^2} = 64{,}03124$, and consequently A a ($=r+n-x$) $= r+n-\sqrt{\frac{rs}{2} + n^2} = 25{,}96876$ Feet, the required Breadth of the Walk.

QUESTION XXIV.

What two Numbers are those, whose Sum is 40, and the Sum of their Cubes 19000?

Put $c=19000$, $s=20$ half the Sum, and $x=$ half the Difference of the required Numbers, then will the Numbers themselves be denoted by $x+s$ and $s-x$, and the Sum of their Cubes will be $\overline{x+s}|^3 + \overline{s-x}|^3$, or $6sx^2 + 2s^3$, which is, by the Question, equal to c, hence $6sx^2 + 2s^3 = c$, therefore $6\,s\,x^2 = c - 2\,s^3$, and consequently $x = \sqrt{\frac{c-2s^3}{6s}} = 5$; therefore $x+s = \sqrt{\frac{c-2s^3}{6s}} + s = 25$, the greater Number, and $s-x = s - \sqrt{\frac{c-2\,s^3}{6s}} = 15$, the less.

QUESTION XXV.

The Difference, and the Difference of the Cubes of two Numbers being given respectively equal to 8 and 2528; to find the Numbers.

Put

Put $c=2528$, $d=4$ half the Difference of the Numbers sought, and $x=$ half their Sum, then will $x+d$ be the greater, and $x-d$ the less Number; and per Question we have $\overline{x+d}|^3 - \overline{x-d}|^3 = c$, or $6dx^2+2d^3=c$, hence $x=\sqrt{\frac{c-2d^3}{6d}} = 10$, therefore $x+d=\sqrt{\frac{c-2d^3}{6d}}+d = 14$, and $x-d=\sqrt{\frac{c-2d^3}{6d}}-d=6$.

QUESTION XXVI.

Find a Number, from the Cube of which, if three Times its Square be subtracted, the Remainder less 1, shall be equal to three Times the Number sought.

Put x for the Number required, then per Question $x^3-3x^2-1=3x$, or $x^3=3x^2+3x+1$; here by adding x^3 to both Sides of the Equation, and extracting the Cube Root, we have $\sqrt[3]{2} \times x = x+1$, or $\sqrt[3]{2} \times x - x = 1$, and $x=\frac{1}{\sqrt[3]{2}-1}$.

QUESTION XXVII.

Find two Numbers whose Product shall be 1075648, and the Surfolid Root of the greater to the Cube Root of the less, as $3\frac{1}{2}$ to 2.

Put x^5 for the greater and y^3 for the less Number, then it will be $x : y :: 3\frac{1}{2} : 2$, hence $2x=3\frac{1}{2}y$, or $4x=7y$, therefore $y=\frac{4x}{7}$, and $y^3=\frac{64x^3}{343}$; and per Question we have $x^5y^3=1075648$, or $y^3=\frac{1075648}{x^5}$, therefore $\frac{64x^3}{343}=\frac{1075648}{x^5}$, hence $x^8=5764801$; and $x=\sqrt[8]{5764801}=7$, therefore $x^5\ (=7^5)=16807$, and $y^3\left(=\frac{64x^3}{343}\right)=64$.

QUESTION

QUESTION XXVIII.

A Company of Men put 1369l. into the Lottery, each Man contributed as many Pounds as there were Men in the Company: How much did each contribute?

Put x for the Pounds that each Man contributed, which were by the Question equal to the Number of Men in Company, then say, if 1 Man pay x Pounds, what will x Men pay? Answer x^2 Pounds, hence per Question $x^2=1369$, and $x=\sqrt{1369}=37$.

QUESTION XXIX.

Required four Numbers in continued Proportion, such that the Sum of the first and fourth shall be 112, and the Product of the other two 432.

Put x and y for the Extremes; then $x+y=112$, and by Geometrical Progression (Case II.) the Product of the two Means is equal to that of the Extremes, therefore we have $xy=432$ by the Question: From the Square of the first Equation take four Times the second, and there will remain $x^2-2xy+y^2=10816$; here, by extracting the Square Root, we have $x-y=104$, this Equation added to, and subtracted from $x+y=112$, gives $2x=216$, or $x=108$, and $y=4$.

In any Geometrical Progression the Square of the second Term divided by the first, gives the third Term; and it is plain, that the Product of the two Means divided by the second Term, also gives the third Term; put z for the second Term, and you will have $\frac{z^2}{4}=\frac{432}{z}$, or $z^3=1728$, and $z=\sqrt[3]{1728}=12$; hence the third Term $\frac{432}{z}$ becomes $\frac{432}{12}$, or 36, therefore the required Numbers are 4, 12, 36, and 108.

Or by making the greater extreme the first Term, we have $\frac{z^2}{108}=\frac{432}{z}$, or $z=\sqrt[3]{46656}=36$, and $\frac{432}{z}=\frac{432}{36}=12$, the two Means as before.

Hence we have this easy Rule for finding two mean Proportionals between two given Numbers.

Multiply the greater Extreme by the Square of the Less, the Cube Root of the Product gives the less Mean, by which, dividing the Product of the two Extremes, the Quotient is the greater Mean.

QUESTION XXX.

Given the Product, and the Sum of the fourth Powers of two Numbers respectively equal to 40 and 4721; to find the Numbers.

Put x for the greater and y for the less Number required; then per Question $xy=40$, and $x^4+y^4=4721$; add twice the Square of the first Equation to, and subtract it from the second Equation, then extract the Square Root of the Sum and Difference, and you will have $x^2+y^2=89$, and $x^2-y^2=39$, the Sum of these two Equations gives $x^2=64$, or $x=8$; and by the first Equation, we have $y=\frac{40}{x}=\frac{40}{8}=5$.

QUESTION XXXI.

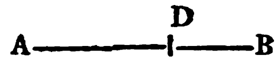
Find two Numbers whose Product shall be p (120), and the Difference of their Squares d (44).

Put x and y for the required Numbers, then $xy=p$, and $x^2-y^2=d$: Add 4 Times the Square of the first Equation to the Square of the second, and the Square Root of the Sum will be $x^2+y^2=\sqrt{d^2+4p^2}$, this Equation added to the second, gives $2x^2=\sqrt{d^2+4p^2}+d$; hence $x=\sqrt{\frac{1}{2}\sqrt{d^2+4p^2}+\frac{1}{2}d}=12$, and $y\left(=\frac{p}{x}\right)=10$.

QUESTION XXXII.

If the right Line AB be divided into two Parts any how in D, and if the Sum of the Segments (AD+DB) be

be multiplied by the whole Line A B, the Product shall be 3422500; Quere, the Length of the Line A B?



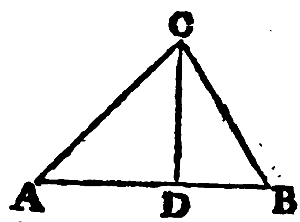
Put $a=3422500$, $AB=x$, and $DB=y$, then will $AD=x-y$, hence per Question we have $AB\times\overline{AD+DB}$ $=a$, or $x\times\overline{x-y+y}=x\times x=x^2=a$, therefore $x=\sqrt{a}$ $=1850$.

By this Operation it appears, that $AB\times\overline{AD+DB}$ $(=x\times\overline{x-y+y}=x^2)=AB^2$ the Square of A B; that is, the Rectangle under a right Line, and the Sum of its Segments is equal to the Square of the whole Line, which is very evident; for the Sum of the Segments $AD+DB$ is equal to the whole Line AB, and any Line or Number multiplied into itself, must necessarily produce the Square of itself.

Hence the 47th Proposition in the first Book of Euclid may be very readily demonstrated; for let A B C be a Triangle right angled at C; from which let fall the Perpendicular C D, then (by Euc. 8. 6.) the Triangles ADC BDC and ABC are similar; therefore

$\left.\begin{array}{l} AB : AC :: AC : AD \\ AB : BC :: BC : DB \end{array}\right\}$ hence

$\left.\begin{array}{l} AC^2=AB\times AD \\ BC^2=AB\times DB \end{array}\right\}$ adding



these two Equations together, we have $AC^2+BC^2=AB\times\overline{AD+DB}$ $(=AB\times AB)=$ AB^2 the Square of the Hypothenuse A B, equal to the Sum of the Squares, of the other two Sides. Q. E. D.

QUESTION XXIII.

Given the Area of a Right-angled Triangle equal to a, to find the Sides in Arithmetical Progression.

Put $2x$ for the greater Leg, y for the common Difference, then will $2x-y$, $2x$ and $2xy+y$ be the Sides of the Triangle

Triangle, hence per Queſtien $2x^2-xy=a$, and per Euc. 47. 1. $\overline{2x+y}|^2=\overline{2x}|^2+\overline{2x-y}|^2$, hence $4x^2=8xy$, or $x^2=2xy$, this Equation taken from twice the firſt, leaves $3x^2=2a$, and $x=\sqrt{\tfrac{2}{3}a}$; but $x^2=2xy$, therefore $x=2y$, and $y=\tfrac{1}{2}x=\tfrac{1}{2}\sqrt{\tfrac{2}{3}a}$; conſequently the required Sides of the Triangle are $2\sqrt{\tfrac{2}{3}a}-\tfrac{1}{2}\sqrt{\tfrac{2}{3}a}$, $2\sqrt{\tfrac{2}{3}a}$ and $2\sqrt{\tfrac{2}{3}a}+\tfrac{1}{2}\sqrt{\tfrac{2}{3}a}$.

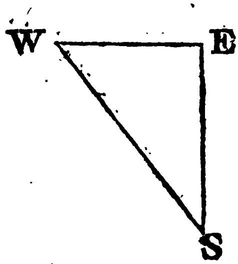
QUESTION XXXIV.

Given the Difference between the Diagonal and Side of a Square, to find its Area.

Put d for the given Difference, $x+d$ for the Side of the Square, then will $x+2d$ be its Diagonal, and per Euc. 47. 1. we have $2\times\overline{x+d}|^2=\overline{x+2d}|^2$, or $x^2=2d^2$, therefore $x=d\sqrt{2}$ hence $\overline{d\sqrt{2}+d}|^2$ is the Area required.

QUESTION XXXV.

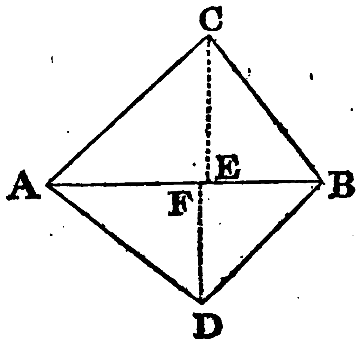
A Ship failed from the Port S, and ſteered between the North and Weſt till her Diſtance failed exceeded her Difference of Latitude by 40, and her Departure by 80 Miles: Quere her Departure, Difference of Latitude and Diſtance failed?



In the Right-angled Triangle, S E W put $x+40=x+a=$ E W then will $x+2a=$ S E and $x+3a=$ S W and per Euc. 47. 1. we have $\overline{x+a}|^2+\overline{x+2a}|^2=\overline{x+3a}|^2$, hence x^2+4a^2, or $x=2a=80$: hence the Departure $x+a$ becomes $3a=120$ Miles $=$ E W the Difference of Latitude $4a=160=$ S E and the Diſtance failed $5a=200$ Miles $=$ S W

QUESTION XXXVI.

In the Trapezium A B C D, are given A C $=28=a$, B C $=26=b$, AD $=24=c$, DB $=18=d$, and AB $=30=e$, to find its Area.



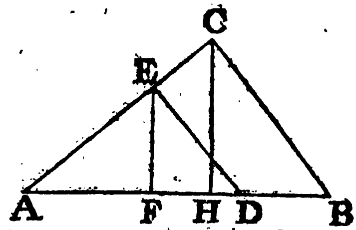
Make C E $=x$ and D F $=y$ each perpendicular to the Diagonal A B; then per Euc. 47. 1. we have A E + E B $= \sqrt{a^2-x^2} + \sqrt{b^2-x^2} = e$, or $e - \sqrt{a^2-x^2} = \sqrt{b^2-x^2}$, and, by squaring both Sides $e^2 - 2e\sqrt{a^2-x^2} + a^2 - x^2 = b^2 - x^2$; hence $\sqrt{a^2-x^2} = \frac{a^2+e^2-b^2}{2e}$; put $\frac{a^2+e^2-b^2}{2e}$ $(=16.8) = m$, then $\sqrt{a^2-x^2} = m$, or $a^2-x^2=m^2$, and $x=\sqrt{a^2-m^2}$; and the Area of the Triangle A B C $=\frac{1}{2}ex=\frac{1}{2}e\sqrt{a^2-m^2}=336$.

In like Manner you will find AF $=\sqrt{c^2-y^2}=\frac{c^2+e^2-d^2}{2e}$ $(=19.2)=n$, and $y=\sqrt{c^2-n^2}$ and the Area of ABD $=\frac{1}{2}ey=\frac{1}{2}e\sqrt{c^2-n^2}=216$; hence $\frac{1}{2}e\sqrt{a^2-m^2} + \frac{1}{2}e\sqrt{c^2-n^2} = \frac{1}{2}e \times \overline{\sqrt{a^2-m^2}+\sqrt{c^2-n^2}} = 552 =$ the Area of the Trapezium ABCD, as required.

QUESTION XXXVII.

Given the Base A B of a Triangular Field AB C, equal to 49.5 Chains, and its Area equal to 40.8375 Acres; to find a Point in the Base from which a right Line D E being drawn

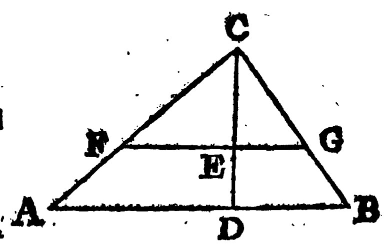
drawn parallel to B C, the Triangle A D E shall contain 21.6 Acres.



Make E F and C H perpendicular to A B, then dividing the given Area 408.375 (in Square Chains) by half the Base 49.5, the Quotient will be $16.5 = CH = p$; put $a = 216$ Square Chains = the Area of ADE; $AB = 49.5 = b$, and $AD = x$, then will $EF = \frac{2a}{x}$, and by similar Triangles it will be $AD : EF :: AB : CH$, that is, $x : \frac{2a}{x} :: b : p$, or $px^2 = 2ab$, hence $x = \sqrt{\frac{2ab}{p}} = 36$.

QUESTION XXXVIII.

Given the Area (a) and Base (b) of a Triangle ABC, to divide it into two equal Parts by a Line F G drawn parallel to the Base A B.



Put $FG = x$, and make C D perpendicular to A B, then will $CD = \frac{2a}{b}$, and $CE = \frac{a}{x}$, and $\frac{a}{x} : x :: \frac{2a}{b} : b$, hence $2x^2 = b^2$, therefore $x = \frac{b}{\sqrt{2}} = \frac{1}{2} b \sqrt{2}$, and $CE \left(= \frac{a}{x}\right) = \frac{2a}{b\sqrt{2}}$.

 QUESTION

QUESTION XXXIX.

There is a hewn Tree 12 Feet long, which is in the Form of the Frustum of a Pyramid, the Side of the Square of its greater Base is 21 Inches, and that of its less is 3 Inches; How far must I measure from the greater End, to cut off 5 solid Feet by a Plane parallel to the Base.

Let the Tree be denoted by ABHK, and complete the Pyramid ABC, then as 21—3, or 18 : 12 : : 21 : 14 Feet $=168$ Inches $=$ CD; hence $21\times21\times168\div3=24696=$ the Solidity of the whole Pyramid ABC, and $24696-1728\times5=16056=$ the Solidity of FGC: Put $a=168=$ CD, $s=24696$, $d=16056$, and $x=$ CE; then since all Pyramids (of the same Base) are as the Cubes of their Altitudes, it will be $s : a^3 : : d : x^3$, hence $sx^3=a^3d$, or $x=a\sqrt[3]{\frac{d}{s}}=145.5390478$, consequently, $a-a\sqrt[3]{\frac{d}{s}}=22.460952$ Inches $=$ DE, the Length required.

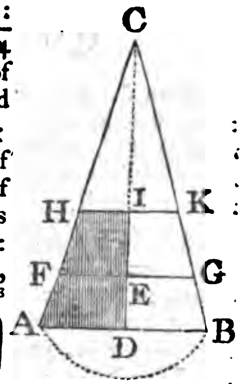
QUESTION XL.

Given the Top and Bottom Diameters of the Frustum (ABHK) of a Cone $=25$ and 40 Inches, with its Height DI$=48$; which is to be divided into two equal Parts, by a Plane parallel to its Base.

Required the Diameter where cut off, and the Length of each Part.

First,

First, as 40—25, or $\overline{15 : 48}$ :: 25 : 80 = CI; then $\overline{25 \times 25 \times .7854 \times 80} \div 3 = 13090$ = the Solidity of the Conic Section H K C, and $\overline{25 \times 40 + \overline{25}|^2 + \overline{40}|^2} \times .7854 \times 48 \div 3 = 40526{,}64$ = the Solidity of the Fruſtum A B H K; the Half whereof added to 13090, gives 33353·32 = the Solidity of F G C: Put $a=80$, $c=13090$, $s=33353{\cdot}32$, and C E $=x$; then it will be $c : a^3 :: s : x^3$, or $cx^3 = a^3 s$, hence $x = a\sqrt[3]{\frac{s}{c}}$

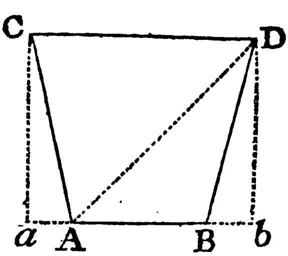


$= 109.267195$, and $a\sqrt[3]{\frac{s}{c}} - a = 29{,}267195$ = E I, conſequently, D F = 18.732805, and $a : 25 :: a\sqrt[3]{\frac{s}{c}} : 25\sqrt[3]{\frac{s}{c}} = 34.1459984325$ = F G, the Diameter required.

QUESTION XLI.

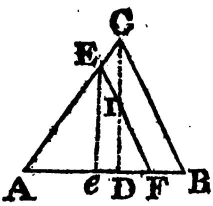
Given the Top Diameter, the Depth and Diagonal of a Tub equal to 96, 60 and 100 Inches reſpectively; to find its Content in Ale Gallons.

Let A B C D repreſent the Tub, whoſe Diagonal A D = 100 Inches $=d$; from D, on A B produced, let fall the Perpendicular D b = C a = 60 Inches $= p$ the Depth, and put A $b = x$, then per Euc. 47. 1. we have $x^2 + p^2 = d^2$, or $x = \sqrt{d^2 - p^2}$; and (2 A b—CD =) $2\sqrt{d^2 - p^2} - 96 = 64$ Inches = A B the Bottom Diameter; hence $\overline{96 \times 64 + \overline{96}|^2 + 64^2} \times .7854 \times 20 \div 282 =$ 1083.74 the Content in Ale Gallons.



QUESTION XLII.

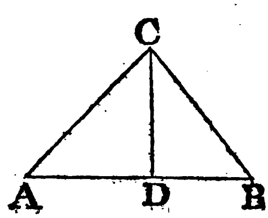
Given the Sides of the Triangle A B C, namely A C $=15$, B C $=13$, and A B $=14$; draw the Line E F parallel to B C, so as to cut the Perpendicular C D exactly in the Middle as at I. Required the Area of the Segment F E C B.



First, A B : A C+C B :: A C−C B : $4=$ A D−DB, hence A D$=9$, and D B$=5=s$; putting B C$=13=a$, and C D$=x$, we have $x^2+s^2=a^2$ (per Euc. 47.1) and x, or C D $=\sqrt{a^2-s^2}=12$, and $\sqrt{a^2-s^2} : s :: \frac{1}{2}\sqrt{a^2-s^2} : \frac{1}{2}s=2\frac{1}{2}=BF=$DF; also A B : A F :: A C : $12\frac{9}{28}=$ A E, and A C : C D :: A E : $9\frac{6}{7}=$ Ee, a Perpendicular let fall on A B from E, consequently, $\frac{1}{2}$ A B × CD−$\frac{1}{2}$ A F × Ee $=7\times12-5\frac{3}{4}\times9\frac{6}{7}=27\frac{9}{28}=$ the Area of F E C B, as required.

QUESTION XLIII.

In the Triangle A B C, C D being let fall from the Right Angle C, perpendicular to A B; there is given A C−B C$=5$, and A B − C D$=13$; to determine the Triangle.



Put $a=5$, $d=13$, then will C D$=\sqrt{d^2-a^2}=12=p$, hence A B$=12+13=25=b$, and putting A C$=x+\frac{1}{2}a$, we have B C $=x-\frac{1}{2}a$; and $b : x+\frac{1}{2}a :: x-\frac{1}{2}a : p$, therefore $x^2-\frac{1}{4}a^2==bp$, or $x=\sqrt{\frac{1}{4}a^2+bp}$: hence A C $=\sqrt{\frac{1}{4}a^2+bp}+\frac{1}{2}a=20$, and B C $=\sqrt{\frac{1}{4}a^2+bp}-\frac{1}{2}a=15$.

QUESTION XLIV.

A Malster has a square Kiln whose Area is 324 Feet, but he intends to pull it down and built another of the same

ſame Form whoſe Area ſhall be 2.25 Times as much: Quere the Length of its Side?

Put $a=324$, $b=2.25$; and $x=$ the Side ſought, then will $x^2=ab$, the Area of the new Kiln, and $x=\sqrt{ab}=27$ Feet, its Side.

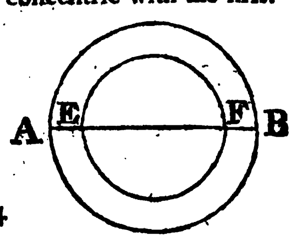
QUESTION XLV.

It is required to plant 2420 Trees on 12 Acres of Land, the Trees are to be ſet at equal Diſtances in Rows parallel to one another, and the Width of each Space between the Rows, is to be to the Diſtance between Tree and Tree in each Row, as 3 to 2: How far muſt theſe Trees ſtand aſunder.

Let $a=58080$ the Square Yards in 12 Acres, $t=2420$, $m=3$, $n=2$; put mx and nx for the required Diſtances in Yards; then will $mx\times nx$, or mnx^2 be the Area which one Tree takes, and $mntx^2$ is the Area which they all take; hence per Queſtion, $mntx^2=a$, and $x=\sqrt{\frac{a}{mnt}}$, hence mx, or $m\sqrt{\frac{a}{mnt}}=6$ Yards, the Diſtance between Row and Row, and $n\sqrt{\frac{a}{mnt}}=$ Yards, the Diſtance between Tree and Tree in each Row.

QUESTION XLVI.

It is required to find the Diameter of a Circle that ſhall contain a given Area, and to divide the Area into two equal Parts, by another Circle concentric with the firſt.



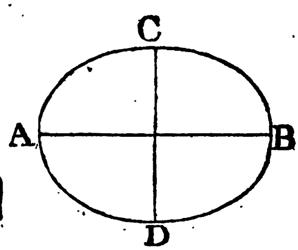
Suppoſe the Area be an Acre of Land, equal to 43560 ſquare Feet, for which put a, let $c=,7854$; and $x=AB$ the Diameter of the Circle containing an Acre, then will $cx^2=a$, therefore $x=\sqrt{\frac{a}{c}}=235,504$

 Feet

Feet, and consequently $EF=\sqrt{\frac{a}{2c}}=166.526$ Feet, the Diameter of the Circle containing half an Acre.

QUESTION XLVII.

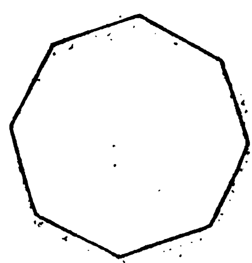
Find the Diameters (in the Proportion of 8 to 5) of an Ellipsis that shall contain a given Area.



Put $a=$ the proposed Area, $c=,7854$; and the Transverse Diameter $AB=8x$, then will the Conjugate Diameter $CD=5x$; and $8x\times5x\times c$, or $40cx^2=a$, therefore $8x=8\sqrt{\frac{a}{40c}}$, and $5x=5\sqrt{\frac{a}{40c}}$.

QUESTION XLVIII.

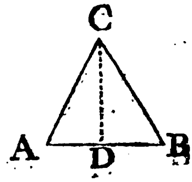
Find the Side of an Octagon that shall contain four Acres.



Put $a=174240$ the Square Feet in 4 Acres, $s=4,828427$, and $x=$ the Side sought, then will $sx^2=a$, and $x=\sqrt{\frac{a}{s}}=189.9639$ Feet $=63.3213$ Yards.

QUESTION XLIX.

Find the Side of an Equilateral Triangle that shall comprize a given Area.



Put $a=$ the given Area; $AC=CD=AB=x$, and make CD perpendicular to AB; then will $CD=\sqrt{x^2-\frac{1}{4}x^2}=\frac{1}{2}x\sqrt{3}$, this multiplied by half the Base x, gives $\frac{1}{4}x^2\sqrt{3}=a$, hence $x=\frac{2\sqrt{a}}{\sqrt[4]{3}}$.

SECTION XVI.

Of Adfected Quadratic Equations.

AN Equation that involves an unknown Quantity, and at the same Time the Square of that Quantity is called an Adfected Quadratic: Of these Equations there are three Forms, each of which contains two different Roots or Values of the unknown Quantity, which Equations may be produced as follow:

CASE I.

Suppose $x=a=10$, and $x=b=6$; then will $x-a=0$, and $x-b=0$: Multiply the Equations $x-a=0$, and $x=b=0$, together, and you will have $\overline{x-a}\times\overline{x-b}=0$, or $x^2-\overline{a+b}\times x+ab=0$, a Quadratic Equation, containing two different Roots, in which it is obvious that $(a+b)$ the Co-efficient of the second Term $-\overline{a+b}\times x$, is the Sum of the two Roots (a and b) having their Signs changed, and the third Term $a\,b$, is the Product of both the Roots multiplied together.

Put the Sum of the Roots $a+b$ $(=16)=2s$, and their Product $a\,b$ $(=60)=p$, then will the Equation $x^2-\overline{a+b}\times x+ab=0$, become $x^2-2sx+p=0$, or $x^2-2sx=-p$, an Equation of the first Form.

CASE II.

If one Root be affirmative, as $x=a=10$, and the other negative, $x=-b=-6$; then will $x-a=0$, and $x+b=0$, these two Equations multiplied together, give $x^2-ax+bx-ab=0$, or $x^2-ax+bx=ab$.

Put $-2d=-a+b$ $(=-10+6)=-4$, and $p=ab$, as before, then will $x^2-ax+bx=ab$, become $x^2-2dx=p$, an Equation of the second Form.

Here, because a is greater than b, therefore $-a+b$ is (-4) negative, and therefore the second Term $(-2dx)$

is

is negative; but if b had been greater than a, it is evident then that the second Term of the last Equation would have been ($+2dx$) affirmative.

CASE III.

If both the Roots are negative, namely $x=-a=-10$, and $x=-b=-6$; then will $x+a=0$, and $x+b=0$, and these two Equations multiplied together, give $x^2+ax+bx+ab=0$, or $x^2+sx+p=0$, or $x^2+sx=-p$, an Equation of the third Form.

CASE IV.

These Equations may be all solved by the following general

RULE.

Add to both Sides of the Equation the Square of half the Co-efficient of the second Term, and the Side that involves the unknown Quantity will be a complete Square; extract the Square Root from both Sides of the Equation, which you will find, on one Side, always to be the unknown Quantity with half the foresaid Co-efficient connected to it; so that by transposing this half you will obtain the Value of the unknown Quantity expressed in known Terms.

Thus, let it be required to solve the Equation $x^2+2sx=-p$: Here half the Co-efficient of the second Term ($-2sx$), is $-s$, whose Square is s^2, which being added to both Sides of the Equation $x^2-2sx=-p$, it becomes $x^2-2sx+s^2=s^2-p$, hence by extracting the Square Root we have $x-s=\sqrt{s^2-p}\,(\sqrt{64-60}=\sqrt{4})=2$; therefore $x=s+\sqrt{s^2-p}\,(=8+2)=10$, the greater Root, and $s-\sqrt{s^2-p}=8-2=6$, is the less Root.

Hence you may observe, that the Sum of the radical Quantity, and half the Co-efficient of the second Term is the greater Root a; and half that Co-efficient minus the radical Quantity is the less Root b. But these Roots may

may be expressed thus, $x=s\pm\sqrt{s^2-p}$, where the upper Sign (+) gives the greater; and the lower (−) the less Root of the Equation.

N. B. Adding the Square of half the Co-efficient of the second Term to both Sides of the Equation, is called completing the Square.

And the Equation $x^2-2dx=p$, being solved as that above, gives $x=d\pm\sqrt{d^2+p}=2\pm8=10$, or to -6.

Again, from the Equation $x^2+2sx=-p$, by completing the Square, extracting the Root, &c. we get $x=-s\pm\sqrt{s^2-p}=-8\pm2=-10$, and to -6. Here because $-s$ is negative, therefore the lower Sign (−) gives the greater Root.

CASE V.

When known Quantities are mixed with unknown ones, transpose all the known Terms to one Side, and the unknown ones to the other Side of the Equation; if the highest Power of the unknown Quantity be multiplied by any Co-efficient (except Unity), divide all the Terms by that Co-efficient, and if the Sign of the highest Power be negative, change the Signs of all the Terms of the Equation before you complete the Square.

Thus, for Instance, let it be required to solve the Equation $3x^2+21x-35=8x^2-19x-275$; here, by Transposition, we have $-5x^2+40x=-240$, and dividing by 5, we get $-x^2+8x=-48$, this Equation, by changing the Signs, becomes $x^2-8x=48$, hence by completing the Square, we have $x^2-8x+16$ $(=16+48)=64$, and extracting the Root, we get $x-4=\sqrt{64}=8$, therefore $x=4+8=12$, the other Value of x, is $4-8=-4$.

And because every Quadratic Equation contains two Roots, therefore in the Solution of Problems, producing Quadratics, you must observe to take that Root which will answer the Conditions of the Question.

CASE

CASE VI.

When an Equation is prepared for completing the Square if the known Term be negative, and greater than the Square of half the Co-efficient of the second Term, then the Roots will be imaginary or impossible: for

EXAMPLE I.

Suppose $x^2-6ax=-13a^2$, then by completing the Square, we have $x^2-6ax+9a^2=9a^2-13a^2$, or $x^2-6ax+9a^2=-4a^2$: and by extracting the Root, we get $x-3a=\pm\sqrt{-4a^2}$ therefore $x=3a\pm\sqrt{-4a^2}$: where the two Values of x are imaginary or impossible, because there is no Quantity either affirmative or negative, which will, when multiplied into itself, produce $-4a^2$, and therefore the Square Root of $-4a^2$ cannot be assigned. And though imaginary Roots may sometimes come into Use, yet when derived from the Solution of Problems, they generally denote that the Thing proposed is impossible.

The Roots of Quadratic Equations are always either both possible, or both impossible together.

CASE VII.

All Equations, whatever, in which there are only two different Dimensions of the unknown Quantity, if the Exponent of one be just twice that of the other, may be solved by completing the Square, &c. as those above.

Thus, let it be proposed to solve the Equation $x^{2n}-ax^n=b$: Here half the Co-efficient of the second Term is $\frac{a}{2}$ whose Square $\frac{a^2}{4}$, being added to both Sides of the Equation, it becomes $x^{2n}-ax^n+\frac{a^2}{4}=\frac{a^2}{4}+b$; whence by extracting the Square Root we have $x^n-\frac{a}{2}=\pm\sqrt{\frac{a^2}{4}+b}$, and therefore $x^n=\frac{a}{2}\pm\sqrt{\frac{a^2}{4}+b}$;

hence

hence by extracting the n^{th} Root we get $x = \sqrt[n]{\frac{a}{2} \pm \sqrt{\frac{a^2}{4} + b.}}$

Here follows a Collection of Questions with their Solutions, which will further illustrate the Rule for Quadratic Equations.

QUESTION I.

Find that Number to which if you add 12, and multiply the Sum by the Number required, the Product shall be 589.

Put x for the Number sought, then per Question, we have $\overline{x+12} \times x = 589$, or $x^2+12x=589$; and by completing the Square, we have $x^2+12x+36=625$, hence, by extracting the Root, we get $x+6=\sqrt{625}=25$, therefore $x=25-6=19$.

QUESTION II.

A and B set out from the same Place, and both travel the same Way, A goes one Mile the first Day, two the second, three the third, &c. in Arithmetical Progression; B sets out five days after A, and travels twelve Miles every Day; how long and how far will A travel before he is overtaken by B.

It is evident when the first Term and common Difference are both Unity, that the Units in the last or greatest Term will be equal to the Number of Terms: Put x for the Number of Terms or Days travelled by A, then will $x-5$ denote the Days B travelled, which at twelve Miles a Day, gives $12x-60$, for the Miles travelled by B: Again, $x+1$, is the Sum of the Extremes, or Miles travelled in the first and last Day by A, which being multiplied by $\frac{x}{2}$, half the Number of Terms, gives $\frac{x^2+x}{2}$ Miles, for the Distance A travelled before he was overtaken

taken by B: And since they both set out from the same Place, it is plain when B overtook A, that they had each travelled the same Number of Miles, whence we have this Equation $\frac{x^2+x}{2}=12x-60$; hence we get $x^2-23x=-120$; here by completing the Square and extracting the Root, we have $x-11,5=\pm\sqrt{12,25}$, therefore $x=11,5\pm\sqrt{12,25}=8$ Days; and the Distance travelled is $12x-60=36$ Miles.

QUESTION III.

There are four Numbers in Harmonic Proportion, whose Sum is 55, and if the second Term be taken from twice the first, the Remainder will be 2; but if half the fourth Term be added to twice the first, the Sum will be 25: Required those Numbers?

Put n, x, y, and z respectively for the Numbers sought,

Then per Question $\left\{\begin{array}{r} n+x+y+z=55 \\ 2n-x=2 \\ 2n+\frac{1}{2}z=25 \end{array}\right\}$:

And by Harmonic Proportion, it will be $n-x:y-z::n:z$, hence we have $nz-xz=ny-nz$, or $ny=2nz-xz$: From the second Equation we get $x=2n-2$, and from the third $z=50-4n$; write $2n-2$ for x, and $50-4n$ for z, in the Equation $ny=2nz-xz$, and you will have $ny=100-8n$, or $y=\frac{100-8n}{n}$: and by writing $2n-2$, $\frac{100-8n}{n}$ and $50-4n$, respectively for x, y, and z, in the first Equation, there will arise $n+2n-2+\frac{100-8n}{n}+50-4n=55$, or $n^2+15n=100$; whence by completing the Square, and extracting the Root, we get $n+7,5=\sqrt{156,25}$, hence $n=\sqrt{156,25}-7,5=5$; therefore $x\ (=2n-2)=8$, $y\left(=\frac{100-8n}{n}\right)=12$, and $z\ (=50-4n)=30$.

QUESTION

QUESTION IV.

A Gentleman has a Garden in the Form of a Right-angled Parallelogram, whose Length to its Breadth is as 6 to 1; but finding it convenient, intends to inclose twice the Area with the same Quantity of walling: Quere the Proportion of the Sides of the Garden when its Area is so increased?

It is plain, that the nearer the Length and Breadth of a Parallelogram approach to an Equality, the greater its Area will be, and because the Perimeter must, (by the Question) be the same as when its Area was but half as much, therefore it is evident that its Breadth must be increased just so much as its Length is decreased: Let $6-x$, and $x+1$ represent the required Ratio, then the Product of these two Quantities must be equal to twice that of (6 and 1) the two Numbers denoting the given Ratio, for these Products must necessarily be in the same Proportion to one another as the Areas, namely, as 2 to 1, hence we have $\overline{6-x}\times\overline{x+1}=6\times1\times2$, or $-x^2+6x-x+6=12$, therefore $x^2-5x=-6$, whence, by completing the Square, &c. we get $x=2,5-\sqrt{,25}=2$: therefore $6-x=6-2=4$, and $x+1=3$; so that the Length to its Breadth, must be as 4 is to 3.

QUESTION V.

There are two square Numbers whose Sum is 400, and they are reciprocally proportional to the square Numbers 64, and 576: What are those Numbers?

Put $a=64$, $b=576$, $s=400$; and $x=$ one of the square Numbers required, then will the other be $s-x$, and by the Question it will be $a:b::x:s-x$, hence by reciprocal Proportion, we get $\frac{ab}{x}=s-x$, or $ab=sx-x^2$, therefore $x^2-sx=-ab$, here, by completing the Square, and extracting the Root, we have $x-\frac{s}{2}=\pm\sqrt{\frac{s^2}{4}-ab}$,

therefore

therefore $x = \frac{s}{4} \pm \sqrt{\frac{s^2}{4} - ab} = 256$, or 144; and these two Roots will answer the Conditions of the Question; for they are square Numbers, their Sum is 400, and 64 : 576 :: 256 : 144.

The above Operation (with respect to square Numbers) is founded on this evident Principle, that if any square Number be multiplied or divided by a square Number, the Product and Quotient will each of them be a square Number; thus the Product ab, being divided by the assumed square Number x, the Quotient $\frac{ab}{x}$ is a square and therefore the fourth Number $s-x$ must be a Square; because $\frac{ab}{x} = s-x$.

QUESTION VI.

Given two square Numbers, namely 25 and 144; to find two other Reciprocals, whose Difference shall be equal to a third given square Number, namely 64.

Put $a=25$, $b=144$, $d=64$, and $x=$ the less Number sought, then will $x+d$ be the greater, and per Question it will be recip. $a : b :: x+d : x$, hence we get $\frac{ab}{x+d} = x$, or $x^2 + dx = ab$, and by completing the Square, &c. we have $x + \frac{d}{2} = \sqrt{ab + \frac{d^2}{4}}$, therefore $x = \sqrt{ab + \frac{d^2}{4}} - \frac{d}{2} = 36$, and $(x+d=)\ \sqrt{ab + \frac{d^2}{4}} + \frac{d}{2} = 100$, are the two Numbers required.

QUESTION VII.

Divide 506 into two such Parts, that the greater may be equal to the Square of the less.

Put

Put $a=506$, and $x=$ the leſs Part required, then will $a-x$ be the greater Part, and per Queſtion we have $x^2=a-x$, or $x^2+x=a$; here the Co-efficient of the ſecond Term is Unity, the half whereof being ſquared and added to both Sides of the Equation, it becomes $x^2+x+,25=a+,25$, hence we get $x+,5=\sqrt{a+,25}$, therefore $x=\sqrt{a+,25}-,5=22$, and $a-x$, or $a-\sqrt{a+,25}+,5=484$.

QUESTION VIII.

A Perſon has now due to him 400l. and 7 Years hence 360l. more will be due to him from the ſame Debtor, they agree to make but one Payment of the Whole, and that without Loſs on either Side, allowing ſimple Intereſt at five per Cent. per Annum: The Time of Payment is required?

It is evident that the Debtor muſt retain the 400l. immediately due, till its Intereſt becomes equal to the Diſcount of 360l. for the Time it muſt be paid before due. Put $a=100$, $b=360$, $c=20$, the Intereſt of 400l. for one Year, $r=5$ the Rate per Cent, $t=7$ Years, and $x=$ the Time ſought, then will $t-x$ be the Time that the 360l. muſt be paid before it is due; and 1 Year : c :: x : cx, the Intereſt of 400l. for the Time x; again 1 Year : r :: $t-x$: $rt-rx$, the Intereſt of 100l. for the Time $t-x$; and by the Rule for Diſcount it will be $a+rt-rx$: $rt-rx$ :: b : $\frac{brt-brx}{a+rt-rx}$ the Diſcount of 360l. for the Time $t-x$, which muſt be equal to cx the Intereſt of 400l. for the Time x; hence we have this Equation $cx=\frac{brt-brx}{a+rt-rx}$, which multiplied by $a+rt-rx$, it becomes $acx+crtx-crx^2=brt-brx$, whence by Tranſpoſition and dividing by cr, we get $x^2-\frac{a}{r}x-\frac{b}{c}x-tx=-\frac{bt}{c}$; or $x^2-\overline{\frac{a}{r}+\frac{b}{c}+t}\times x=-\frac{bt}{c}$; put $s=\frac{a}{r}+\frac{b}{c}+t$ $(=45)$,

($=45$), and you will have $x^2-sx=-\frac{bt}{c}$, here by completing the Square, &c. we get $x=\frac{s}{2}-\sqrt{\frac{s^2}{4}-\frac{bt}{c}}$ $=3$ Years.

QUESTION IX.

In what Time will a Rent of 80l. per Annum amount to 540l. at 5 per Cent. simple Interest?

Put $a=540$, $u=80$, $r=,05$ the Interest of 1l. per Annum, and $t=$ the Time required; then will tu be the Rent due in the Time t, and 1l. : r :: u : ru the Interest due at the End of the second Year, and because there will then be two Years Rent due, therefore the third Year's Interest will be $2ru$; that of the fourth Year will be $3ru$ (the Interest of three Years Rent), the Interest of the fifth Year will be $4ru$, &c. here you see that the common Difference of 1, 2, 3, 4, &c. the Numeral Co-efficients of the Terms ru, $2ru$, $3ru$, $4ru$, &c. is Unity, and since the Interest does not commence till the second Payment, therefore $t-1$ is the Co-efficient of the last Term, and consequently $\overline{t-1}\times ru$ is the last Year's Interest, and because the first Year's Interest is 0, therefore (by Case II. Arith. Prog.) the last Term $\overline{t-1}\times ru$ multiplied by $\left(\frac{t}{2}\right)$ half the Number of Terms, gives $\frac{t}{2}\times\overline{t-1}\times ru$, or $\frac{t^2-t}{2}\times ru$ for the Sum of all the Interest, which being added to tu, the whole Rent, the Sum must evidently be equal to a, the Amount; hence we have this Equation $\frac{t^2-t}{2}\times ru+tu=a$, therefore $\overline{t^2-t}\times ru+2tu=2a$, and $t^2-t+\frac{2t}{r}=\frac{2a}{ru}$; put $s=\frac{2}{r}-1$ ($=39$), then will $t^2+st=\frac{2a}{ru}$, whence by completing the Square, &c. we get

$t+$

$t+\frac{s}{2}=\sqrt{\frac{2a}{ru}+\frac{s^2}{4}}$, therefore $t=\sqrt{\frac{2a}{ru}+\frac{s^2}{4}}-\frac{s}{2}=6$ Years.

Note. If the Payments are Half-yearly, put $u=$ half the Salary, Annuity, &c. and $r=$ half the Intereſt of 1l. for a Year; if Quarterly, put $u=$ one fourth of the Annuity, and $r=$ one fourth of the Intereſt of 1l. per Annum, and t will be equal to the Number of thoſe Half-yearly, or Quarterly Payments.

QUESTION X.

For Example, if an Annuity of 150l. per Annum, payable Half-yearly, amounts to 834,375l. at 5 per Cent. what Time was the Payment forborne?

Here $a=834{,}375$, $u=\frac{150}{2}=75$, $r=\frac{{,}05}{2}={,}025$, and $s=\frac{2}{r}-1=79$, then $t=\sqrt{\frac{2a}{ru}+\frac{s^2}{4}}-\frac{s}{2}=\sqrt{\frac{1668{,}75}{1{,}875}+\frac{6241}{4}}-\frac{79}{2}=49{,}5-39{,}5=10$ Half-years, or 5 Years, the Time required.

QUESTION XI.

Again, if a yearly Penſion of 150l. payable Quarterly, amounts to 839,0625, at 5 per Cent. what was the Time of Forbearance?

Here $a=839{,}0625$, $u=\frac{150}{4}=37{,}5$, $r=\frac{{,}05}{4}={,}0125$, and $s=\frac{2}{r}-1=159$, hence $t=\sqrt{\frac{2a}{ru}+\frac{s^2}{4}}-\frac{s}{2}=\sqrt{\frac{1678{,}125}{0{,}46875}+\frac{25281}{4}}-\frac{159}{2}=\sqrt{9900{,}25}-79{,}5=20$ Quarters, or 5 Years.

 QUESTION

QUESTION XII.

What Time may a Penſion of 40l. per Annum be bought for 245l. at 4 per Cent?

Put $p=245$, $u=40$, $r=,04$, and $t=$ the Time ſought; then will rt be the Intereſt of 1l. for the Time t, and 1l. : rt :: p : rtp the Intereſt of the Principal p, for the ſame Time t, but the Principal and Intereſt added together, gives the Amount, therefore $p+rtp$ is the Amount; and by the foregoing Operation, $\frac{t^2-t}{2}\times ru+tu$ alſo equals the Amount, and Quantities that are equal to one and the ſame Thing, are evidently equal to each other, therefore $\frac{t^2-t}{2}\times ru+tu=p+rtp$, hence $t^2-t+\frac{2t}{r}-\frac{2tp}{u}=\frac{2p}{ru}$; put $s=\frac{2}{r}-\frac{2p}{u}-1$ $(=36,75)$ then will $t^2+st=\frac{2p}{ru}$, here, by completing the Square, &c. we get $t=\sqrt{\frac{2p}{ru}+\frac{s^2}{4}}-\frac{s}{2}=7$ Years.

This and the preceding Theorem may be frequently found, with their Applications, in Arithmetical Books compiled for the Uſe of Schools; therefore I ſhall purſue them no further here; but as the Derivations and Demonſtrations of theſe Theorems are generally omitted in thoſe Books, the above Solutions may therefore ſerve to convince thoſe Arithmeticians into whoſe Hands this may fall, how indiſpenſably neceſſary it is to be acquainted with Algebra, in order to underſtand the Grounds of the Solutions of thoſe Problems, in which the Extraction of Roots is required.

QUESTION

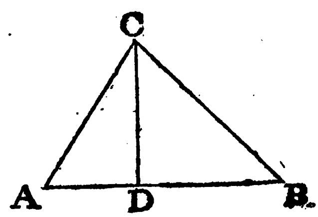
QUESTION XIII.

A Gentleman has a Rectangular Garden whose Area is 12100 square Yards; aud its Breadth, Length, and Diagonal are in Geometrical Progression: Quere, these Dimensions?

Put x for the Breadth of the Garden, in Yards, and r for the Ratio of the Progression, then will $r\,x$ be the Length and r^2x the Diagonal; hence per Question $rx^2 = 12100$, therefore $x = \sqrt{\frac{12100}{r}}$; and per Eu. 47. 1. $r^4x^2 = x^2 + r^2x^2$, dividing by x^2, we have $r^4 = 1 + r^2$, or $r^4 - r^2 = 1$, and by completing the Square, &c. we get $r^2 - \frac{1}{2} = \sqrt{1{,}25}$, therefore $r = \sqrt{\frac{1}{2} + \sqrt{1{,}25}} = 1.272019$, hence $x = \sqrt{\frac{12100}{r}} = 97{,}53167$, $rx = 124.06226$ and $r^2x = 157.80955$.

QUESTION XIV.

Two Ships sailed from the Port C; the first of them in 60 Miles arrived at her intended Port A, which is in the same Latitude with the Port B, to which the second Ship sailed; they find by Calculation, that their Distances sailed are mean Proportionals between the Sum of their Departures and the Difference of their Latitude; the less Mean of this Progression being A C=60; the Rest of the Terms are required?



Make C D perpendicular to A B, put A C $= 60 = a$, and B C $= x$, the two Means, then the Extremes will be C D $= \frac{a^2}{x}$, and AB $\frac{x^2}{a}$, and per Eucl. 47. 1. we have A D+DB $= \sqrt{a^2 - \frac{a^4}{x^2}}$

$+\sqrt{x^2-\frac{a^4}{x^2}}=\frac{x^2}{a}$, or $x^3-a^2\sqrt{x^2-a^2}=a\sqrt{x^4-a^4}$ and by squaring both Sides, $x^6-2a^2x^3\sqrt{x^2-a^2}+a^4x^2-a^6=a^2x^4-a^6$, hence $x^4-a^2x^2+a^4=2a^2x\sqrt{x^2-a^2}$; here by squaring again, and transposing, we get this Equation $x^8-2a^2x^6-a^4x^4+2a^6x^2+a^8=0$, whose square Root is $x^4-a^2x^2-a^4=0$, therefore $x^4-a^2x^2=a^4$, here, by completing the Square, &c. we have $x^2-\frac{1}{2}a^2=\sqrt{\frac{5}{4}a^4}=\frac{1}{2}a^2\sqrt{5}$ and $x=a\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{5}}=76.3211697$ Miles, the Distance failed by the second Ship: hence $\frac{x^2}{a}=97.0820157$, the Sum of their Departures, and $\frac{a^2}{x}=47.1690883$, their Difference of Latitude, as required.

But since $AB:BC::AC:CD$, it is well known that the Angle ACB is a right one, therefore by Euc. 47. 1. we have $\frac{x^4}{a^2}=x^2+a^2$, or $x^4-a^2x^2=a^4$, and consequently $x=a\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{5}}$, as before.

QUESTION XV.

The continual Product of four Numbers in Arithmetical Progression is 1680, and their common Difference is 4; What are those Numbers?

Let $x-6$, $x-2$, $x+2$, and $x+6$, represent the required Numbers, then per Question we have $\overline{x-6}\times\overline{x+6}\times\overline{x-2}\times\overline{x+2}=1680$, or $x^4-40x^2+144=1680$, therefore $x^4-40x^2=1536$, here by completing the Square and extracting the Square Root, we get $x^2-20=\sqrt{1936}=44$, hence $x^2=20+\sqrt{1936}=64$, and consequently $x=\sqrt{20+\sqrt{1936}}=8$, therefore the Numbers required are 2, 6, 10, and 14.

QUESTION XVI.

Given the Sum of two Numbers equal to 20, and the Sum of their fifth Powers equal to 281600, to find the Numbers.

Put $a=281600$, $2s=20$, and $x=$ half the Difference of the two Numbers sought, then will $x+s$ be the greater, and $s-x$ the less Number, and per Question we have $\overline{x+s}|^5 + \overline{s-x}|^5 = a$, or $10sx^4+20s^3x^2+2s^5=a$, here, by Transposition and Division, our Equation becomes $x^4 + 2s^2x^2 = \frac{a}{10s} - \frac{s^4}{5}$, whence by completing the Square &c. we get $x^2+s^2=\sqrt{\frac{a}{10s}+\frac{4s^4}{5}}\,(=104)$, hence $x=\sqrt{\sqrt{\frac{a}{10s}+\frac{4s^4}{5}}-s^2}=2$, therefore 12 and 8 are the required Numbers.

QUESTION XVII.

Find a Number to which if you add its Biquadrate, and from that Sum subtract twice its Cube, the Remainder shall be 1722.

Put $a=1722$, and $x+0{,}5=$ the Number required; then per Question we have $\overline{x+0{,}5}|^4 - 2\times\overline{x+0{,}5}|^3 + x+0{,}5=a$, or $x^4-1{,}5x^2+0{,}3125=a$, hence $x^4-1{,}5x^2=a-0{,}3125$, and by completing the Square, &c. we have $x^2-0{,}75=\sqrt{a+0{,}25}$, therefore $x=\sqrt{0{,}75+\sqrt{a+0{,}25}}=6{,}5$, and consequently $x+0{,}5=\sqrt{0{,}75+\sqrt{a+0{,}25}}+0{,}5=7$.

QUESTION XVIII.

Given the Sum of two Numbers equal to s, and their Product equal to p, to find the Sum of their Squares, Cubes, Biquadrates, &c.

Put x and y for the two Numbers, then per Question $x+y=s$, and $xy=p$: From the Square of the first Equation, take twice the second, and you will have $x^2+y^2=s^2-2p$,

$-2p$, the Sum of their Squares, which multiply by the first Equation, and from the Product $x^3+xy^2+x^2y+y^3=s^3-2sp$, subtract that of the first and second Equations multiplied together, and there will remain $x^3+y^3=s^3-3sp$, the Sum of their Cubes, which multiply likewise by the first Equation, and from the Product $x^4+xy^3+x^3y+y^4=s^4-3s^2p$, subtract the Sum of the Squares multiplied by the second Equation, and you will have $x^4+y^4=s^4-4s^2p+2p^2$, the Sum of the Biquadrates; and thus by multiplying the Sum of the Powers last found by s, and subtracting from the Product the Sum of the next preceding ones multiplied by p, you may obtain the Sum of any Powers proposed; and the Sum of the n^{th} Powers will be $x^n+y^n=$

$$s^n-ns^{n-2}p+n\times\frac{n-3}{2}\times s^{n-4}p^2-n\times\frac{n-4}{2}\times\frac{n-5}{3}$$

$$\times s^{n-6}p^3+n\times\frac{n-5}{2}\times\frac{n-6}{3}\times\frac{n-7}{4}\times s^{n-8}p^4, \text{ \&c.}$$

If $n=5$, then will $x^n+y^n=x^5+y^5=s^5-5s^{5-2}p+5\times\frac{5-3}{2}\times s^{5-4}p^2=s^5-5s^3p+5sp^2$; if $n=6$, then will $x^n+y^n=x^6+y^6=s^6-6s^4p+9s^2p^2-2p^3$.

QUESTION XIX.

There are four Numbers in Geometrical Progression whose Sum is 45, and the Sum of their Squares is 765; Quere those Numbers?

Put $a=45$, $b=765$, $p=$ the Product of the two Means or Extremes; for those Products are equal to each other; and let $s=$ the Sum of the Means, then will $a-s$ be the Sum of the Extremes, and by the last Problem, the Sum of the Squares of the Extremes will be $\overline{a-s}|^2-2p$, and that of the Means s^2-2p, hence per Question $s^2+\overline{a-s}|^2-4p=b$; moreover, s^3-3sp is the Sum of the Cubes of the Means, which, by 8 Geometrical Progression, is equal to the Sum of the Extremes multiplied by their Product, whence $p\times\overline{a-s}=s^3-3sp$, hence we get $p=\frac{s^3}{a+2s}$ this

Value

Value being wrote for p in the foregoing Equation $s^2+\overline{a-s}|^2-4p=b$, there will arise $s^2+\overline{a-s}|^2-\frac{4s^3}{a+2s}=b$, or $a^2-2as^2=2bs+ab$, hence $s^2+\frac{b}{a}s=\frac{a^2-b}{2}$, here half the Co-efficient of the second Term is $\frac{b}{2a}$, which being squared and added to both Sides of the Equation and the Root extracted, we have $s+\frac{b}{2a}=\sqrt{\frac{a^2-b}{2}+\frac{b^2}{4a^2}}$ therefore $s=\sqrt{\frac{a^2-b}{2}+\frac{b^2}{4a^2}}-\frac{b}{2a}=18$, and $p=\frac{s^3}{a+2s}=72$.

Having thus found the Sum of the Means 18, and their Product 72, the Means themselves will, by Problem 29, be readily found to be 6 and 12; then 12 : 6 : : 6 : 3, the less Extreme, and therefore the greater Extreme is 24, so that the required Numbers are 3, 6, 12, and 24.

QUESTION XX.

Find three Numbers in Geometrical Progression whose Product is 216, and the Sum of their Cubes 1009.

Let x, xy and xy^2 be the required Numbers, then per Question $x^3y^3=216$, and $x^3+x^3y^3+x^3y^6=1009$; from the first Equation we get $y^3=\frac{216}{x^3}$, and by writing $\frac{216}{x^3}$ for y^3, in the second Equation, we have $x^3+216+\frac{46656}{x^3}=1009$, hence $x^6-793x^3=-46656$, here by compleating the Square, &c. we have $x^3-396,5=\pm\sqrt{110556,25}$, consequently $x=\sqrt[3]{396,5\pm\sqrt{11556,25}}=4$: But $y^3=\frac{216}{x^3}$

$=\frac{216}{64}=\frac{27}{8}$, therefore $y=\sqrt[3]{\frac{27}{8}}=\frac{3}{2}=1,5$, consequently the Numbers sought are 4, 6, and 9.

Lemma. Any Equation which contains several Terms and the Square of their Sum, may be solved by completing the Square, &c. provided the unknown Quantity be comprized only in that Power and Root; and the Coefficient of the said Root or second Term must be considered to be Unity, except it be multiplied by a given Quantity, though the Root may be composed of many Terms both known and unknown. For want of this Observation, some Authors have substituted other unknown Quantities, instead of those they first assumed, in order to avoid high Equations, which might have been solved as readily with their original unknown Quantities, as by any Substitution whatever.

QUESTION XXI.

A Man playing at Hazard or Dice, won the first Throw just so much Money as he had in his Pocket; the second throw he won the Square Root of what he then had, and five Shillings more; the third Throw he won the Square of all he then had; after which his whole Sum was 112l. 16s. What Money had he when he began to play?

Put x for the Money he had when he began to play, then $2x=$ his Sum after the first Throw, and $\sqrt{2x}+5=$ the Winnings at his second throw, therefore $2x+\sqrt{2x}+5$ $=$his Money after the second Throw, which, being added to the Square of itself, the Sum must by the Question be equal to 2256 Shillings, hence $\overline{2x+\sqrt{2x}+5}|^2+2x+\sqrt{2x}+5=2256$; here by completing the Square, &c. we have $2x+\sqrt{2x}+5,5=\sqrt{2256,25}=47,5$, or $\sqrt{\frac{1}{2}x}=21-x$, hence $\frac{1}{2}x=441-42x+x^2$, therefore $x^2-42,5x=-441$, whence $x=21,25-\sqrt{10,5625}=18$ Shillings.

This

This Solution may be performed with fewer Figures by completing the Square at the Equation $x+\sqrt{\frac{1}{2}x}=21$, by which Means you will have $x+\sqrt{\frac{1}{2}x}+\frac{1}{8}=21\frac{1}{8}=\frac{169}{8}$ or $x^{\frac{1}{2}}+\sqrt{\frac{1}{8}}=\frac{13}{\sqrt{8}}$, hence $\sqrt{8x}=12$, and $8x=144$, therefore $x=18$.

QUESTION XXII.

What two Numbers are those, whose Sum, when added together, is equal to their Product when multiplied together; and this Sum or Product, when added to the Sum of their Squares, makes twelve?

Let a be 12, and put x and y for the two Numbers sought, then per Question $x+y=xy$, and $x+y+x^2+y^2=a$; from the second Equation take twice the first, and you will have $x^2+y^2-x-y=a-2xy$, or $\overline{x+y}|^2-x-y=a$; whence by completing the Square, &c. we get $x+y=0,5+\sqrt{a+0,25}=4$; therefore $xy=4$; from the Square of the Equation $x+y=4$ take four Times $xy=4$, and you will have $x^2-2xy+y^2=0$; whose Square Root $x-y=0$, therefore $x=y$, consequently x and y are each equal to 2.

QUESTION XXIII.

What two Numbers are those, whose Sum, added to the Product of their Multiplication, makes Thirty-four, and the same Sum subtracted from the Sum of their Squares, leaves Forty-two.

Put $a=34$, $b=42$, $x=$ the greater and $y=$ the less Number sought, then per Question $x+y+xy=a$, and $x^2+y^2-x-y=b$; add the second of these Equations to twice the first, and you will have $x^2+2xy+y^2+x+y=2a+b$; hence by completing the Square, &c. we get $x+y=\sqrt{2a+b+0,25}-0,5=10$, this Equation taken from $x+y+xy=a$ ($=34$), leaves $xy=24$; now from the Equations

$x+y$

$x+y=10$, and $xy=24$, you will, by proceding as in the last Question, find $x=6$, and $y=4$. See Problem 29.

QUESTION XXIV.

Given $x^3-7x^{\frac{3}{2}}=8$, to find the Value of x.

Transpose x^3, and square both Sides of the Equation, and you will have $49x^3=64-16x^3+x^6$, or $x^6-65x^3=\pm 64$; hence $x^3-32,5=\sqrt{1056,25-64}$, and consequently $x=\sqrt[3]{32,5+\sqrt{1056,25-64}}=4$.

Or thus, by completing the Square at the given Equation we have $x^3-7x^{\frac{3}{2}}+12,25=20,25$; therefore $x^{\frac{3}{2}}-3,5=\sqrt{20,25}=4,5$, hence $x^{\frac{3}{2}}=8$, or $x^3=64$, and therefore $x=\sqrt[3]{64}=4$.

QUESTION XXV.

There are three such Numbers, that if you add the third to twice the first, and from the Sum subtract the second, the Remainder will be 70; and the Product of the first and third, added to the Square of the second is 800.

Moreover, if the Square of their Sum be added to twice the second Number, minus the first, the Remainder will be 4900: Quere, those Numbers?

Let the Numbers required be denoted by x, y, and z.

Then per Question $\begin{cases} 2x-y+z=70, \\ xz+y^2=800, \\ \overline{x+y+z}|^2-x+2y=4900. \end{cases}$

Adding the first Equation to the third, we have $\overline{x+y+z}|^2+x+y+z=4970$, here by completing the Square, &c. $x+y+z+0,5=\sqrt{4970,25}=70,5$, hence $x+y+z=70$, this Equation taken from the first, leaves $x-2y=0$, therefore $x=2y$, this, subtracted from $x+y+z=70$, leaves $y+z=70-2y$, or $z=70-3y$: Now write $2y$ for x, and $70-3y$

$-3y$ for z, in the second Equation, and there will arise $2y\times\overline{70-3y}+y^2=800$, or $y^2-28y=-160$, whence $y=14+\sqrt{36}=20$, therefore x $(=2y)=40$, and z $(=70-3y)=10$.

QUESTION XXVI.

It is required to find the Value of x in this Equation $\overline{x^2+ax-bc}|^{2n}+\overline{x^2+ax-bc}|^n=s$. Here by compleating the Square, and extracting the Square Root $\overline{x^2+ax-bc}|^n+0,5=\sqrt{s+0,25}$, therefore $x^2+ax-bc=\sqrt[n]{\sqrt{s+0,25}-0,5}$, or $x^2+ax=\sqrt[n]{\sqrt{s+0,25}-0,5}+bc$; hence we get $x=\overline{\sqrt[n]{\sqrt{s+0,25}-0,5}+\frac{a^2}{4}+bc}|^{\frac{1}{2}}-\frac{a}{2}$.

QUESTION XXVII.

Quere, the Area of a Right-Angled Triangle, whose Hypothenuse is x^{3x}, and the two Legs x^{2x} and x^x.

By Euclid's 1. 47. In any Right-angled Triangle, the Square of the Hypothenuse is equal to the Sum of the Squares of the other two Sides, whence, and per Question, we have $x^{6x}=x^{4x}+x^{2x}$, hence $x^{6x}-x^4x=x^{2x}$; this Equation divided by x^{2x}, gives $x^{4x}-x^{2x}=1$; here by completing the Square, &c. we get $x^{2x}=,5+\sqrt{1,25}=1,61803398=$ the greater Leg, therefore $x^x=\sqrt{,5+\sqrt{1,25}}=1,27201964=$ the less Leg; consequently the Hypothenuse x^{3x} $(=\sqrt{x^{4x}+x^{2x}})=\sqrt{\overline{1,61803398}|^2+\overline{1,27201964}|^2}=2,0581709$: And multiplying the greater Leg by half the less, gives $1,029085+$, for the Area required.

QUESTION

QUESTION XXVIII.

Given $\left\{ \begin{array}{l} \sqrt{x^2+y^2}=a \\ y\sqrt{x^2-y^2}=b \end{array} \right\}$, to find x and y.

By squaring the first Equation, we get $x^2+y^2=a^2$, or $x^2=a^2-y^2$; and by writing a^2-y^2 for x^2, in the second Equation, we have $y\sqrt{a^2-2y^2}=b$, or $a^2y^2-2y^4=b^2$, therefore $y^4-\frac{a^2}{2}y^2=-\frac{b^2}{2}$, hence $y=\sqrt{\frac{a^2}{4}+\sqrt{\frac{a^4}{16}-\frac{b^2}{2}}}$: and $x=\sqrt{a^2-y^2}=\sqrt{a^2-\frac{a^2}{4}\mp\sqrt{\frac{a^4}{16}-\frac{b^2}{2}}}$.

QUESTION XXIX.

Suppose $x^{4n}-2x^{3n}+x^n=a$, Quere x.

The given Equation may be expressed thus, $\overline{x^{2n}-x^n}|^2-x^{2n}+x^n=a$; here by completing the Square, and extracting the Square Root, we get $x^{2n}-x^n=\frac{1}{2}\pm\sqrt{a+\frac{1}{4}}$: and, consequently, $x=\frac{1}{2}\pm\sqrt[n]{\frac{3}{4}\pm\sqrt{a+\frac{1}{4}}}$.

QUESTION XXX.

Given $\left\{ \begin{array}{l} x^2y^2-y-x^2y=744=a \\ x^2y+y-x\,y=126=b \end{array} \right\}$, to find x and y.

From the second Equation we get $y=\frac{b}{x^2-x+1}$, this Value of y being written in the Sum of the given Equations we have $\frac{b^2x^2}{\overline{x^2-x+1}|^2}-\frac{bx}{x^2-x+1}=a+b=s$; hence by completing the Square, &c. $\frac{bx}{x^2-x+1}=\frac{1}{2}+\sqrt{s+\frac{1}{4}}=n$, or $bx=nx^2-nx+n$, and $x^2-\overline{\frac{b}{n}+1}\times x=$

-1,

-1, therefore $x = \frac{b}{2n} + \frac{1}{2} + \sqrt{\overline{\frac{b}{2n} + \frac{1}{2}}\big|^2 - 1} = 5$, and $y = \frac{b}{x^2 - x + 1} = 6$.

QUESTION XXXI.

Given $\left\{ \begin{array}{r} x^n + y^n = s \\ xy = p \end{array} \right\}$, to find x and y.

By the ſecond Equation $y = \frac{p}{x}$, this Value of y, ſubſtituted in the firſt Equation gives $x^n + \frac{p^n}{x^n} = s$, or $x^{2n} - sx^n = -p^n$, hence by completing the Square, &c. $x^n - \frac{1}{2}s = \pm\sqrt{\frac{1}{4}s^2 - p^n}$, and $x = \overline{\frac{1}{2}s \pm \sqrt{\frac{1}{4}s^2 - p^n}}\Big|^{\frac{1}{n}}$, whence $y = \frac{p}{x} = \frac{p}{\overline{\frac{1}{2}s \pm \sqrt{\frac{1}{4}s^2 - p^n}}\Big|^{\frac{1}{n}}}$.

QUESTION XXXIII.

In the ſame Manner, if there be given $x^n - y^n = d$, and $xy = p$, you will find $x = \overline{\frac{1}{2}d \pm \sqrt{\frac{1}{4}d^2 + p^n}}\Big|^{\frac{1}{n}}$, and $y = \frac{p}{\overline{\frac{1}{2}d \pm \sqrt{\frac{1}{4}a^2 + p^n}}\Big|^{\frac{1}{n}}}$.

QUESTION XXXIII.

Given $x^{2nx} - 8x^{nx} = 3584$, to find x^x, where n denotes a given Number.

By completing the Square, and extracting the Root, we have $x^{nx} - 4 = \sqrt{3600}$, or $x^{nx} = \sqrt{3600} + 4$; therefore $x^x = \sqrt[n]{\sqrt{3600} + 4} = \sqrt[n]{64}$, as required.

Here

Here the Root x may be readily found by Logarithms; be n what it will, as ſhall be ſhown further on.

If $n=3$, then it is plain $x=2$, for then will $x^x=x^2=\sqrt[x]{64}=\sqrt[3]{64}=\overline{8}|^{\frac{2}{3}}$ and ſince in this Caſe $x^2=\overline{8}|^{\frac{2}{3}}$ therefore $x=\overline{8}|^{\frac{1}{3}}=2$.

QUESTION XXXIV.

Given $x^4-12x^3+44x^2-48x=9009=a$; to find x.

It is plain that the propoſed Equation may be expreſſed thus, $\overline{x^2-6x}|^2+8\times\overline{x^2-6x}=a$, here by completing the Square, &c. we have $x^2-6x+4=\sqrt{a+16}$, or $x^2-6x=\sqrt{a+16}-4$; and $x=\sqrt{5+\sqrt{a+16}}+3=13$.

QUESTION XXXV.

Given $x^4-400x^2+560x=196$, to find x.

By tranſpoſing $-400x^2+560x$, and extracting the Square Root, we have $x^2=20x-14$, or $x^2-20x=-14$, and $x=\sqrt{86}+10$.

QUESTION XXXVI.

Given $x^3-279x-278=0$, to find x.

Here it is obvious that one Root of this Equation is -1, for by writing -1 for x in the given Equation it becomes $-1+279-278=0$, which ſhews that -1 is a Root, for the Terms $-1+279-278$ being added together make (0) nothing, and though this negative Root (-1) will not generally ſolve Problems producing ſuch Equations, yet ſince $x=-1$, we have $x+1=0$, and dividing the given Equation $x^3-279x-278=0$ by $x+1$, it will be reduced to $x^2-x-278=0$, a Quadratic, which ſolved, gives $x=\sqrt{278,25}+,5$.

QUESTION

QUESTION XXXVII.

Given $x^4+6x^3-48x=64$, to find x.

By adding $9x^2+48x$ to both Sides, and extracting the Root, we have $x^2+3x=3x+8$, or $x^2=8$; and $x=\sqrt{8}$ $=2\sqrt{2}$.

How to determine when this Method of solving an affected Biquadratic will succeed, and how to find the Quantity which being added to each of its Sides, will complete the Square, shall be copiously treated of in the Course of this Work.

Here follow a few Equations, which are solved by substituting for the unknown Quantities.

QUESTION I.

Given $\left\{\begin{array}{l}\sqrt[3]{x^2}\times\sqrt{y^3}=2y^2\\ 12\sqrt[3]{x}-\sqrt{y}=22\end{array}\right\}$, to find x and y.

Put $u=\sqrt[3]{x}$, and $z=\sqrt{y}$, then will $u^3=x$, and $z^2=y$; and the given Equations will become $u^2z^3=2z^4$, and $12u-z=22$: Dividing the Equation $u^2z^3=2z^4$, by $2z^3$, we get $z=\frac{u^2}{2}$, this Equation added to $12u-z=22$, gives $12u=22+\frac{u^2}{2}$, or $u^2-24u=-44$, hence $u=12$ $-\sqrt{100}=2$, therefore $z=\frac{u^2}{2}=2$; consequently $x=$ $u^3=8$, and $y\ (=z^2)=4$.

QUESTION II.

Given $\left\{\begin{array}{l}x+y+xy=656=a,\\ x^2y+xy^2=46080=b.\end{array}\right.$

The second Equation may be expressed thus, $\overline{x+y}\times xy=b$; put $s=x+y$, and $p=xy$, then will the Equations become

become $s+p=a$, and $sp=b$; from the Square of the Equation $s+p=a$, take four Times $sp=b$, and you will have $s^2-2sp+p^2=a^2-4b$, hence $s-p=\pm\sqrt{a^2-4b}$ $(=\pm 496)$ this Equation added to, and subtracted from $s+p=a$, gives $s=\frac{1}{2}a\pm\frac{1}{2}\sqrt{a^2-4b}=80$, and $p=\frac{1}{2}a\pm\frac{1}{2}\sqrt{a^2-4b}=576$; therefore $x+y=80$, and $xy=576$; from these two Equations, (by proceeding as above) you will find $x=72$, and $y=8$; and thus you may find any two Numbers by having their Sum and Product given, without completing the Square: But it may be observed at the Equation where the Root is added, that the upper Sign $(+)$ gives the greater and the lower $(-)$ the less Number, therefore the Equation whose Root is extracted, needs not be subtracted, since both the Numbers sought may be determined from that Equation to which it is added.

QUESTION III.

$$\text{Given}\left\{\begin{array}{l} xy^3+x^3y=42500=a, \\ x^4+\ y^4=160625=b. \end{array}\right.$$

Put $s=x^2+y^2$, and $p=xy$; then will the Equations become sp $(=\overline{y^2+x^2}\times xy)=a$, and s^2-2p^2 $(=\overline{x^2+y^2}|^2-2\times\overline{xy}|^2=x^4+y^4)=b$; from $sp=a$, we get $p=\frac{a}{s}$, this Value being wrote for p, in the Equation $s^2-2p^2=b$, there arises $s^2-\frac{2a^2}{s^2}=b$, hence $s=\sqrt{\frac{b}{2}+\sqrt{2a^2+\frac{b^2}{4}}}$ $=425$; therefore $p=\frac{a}{s}=100$; now we have $x^2+y^2=425$, and $xy=100$, hence $2xy=200$, this Equation being added to, and subtracted from $x^2+y^2=425$, and the Square Root of the Sum and Difference extracted, we have $x+y=\sqrt{625}=25$, and $x-y=\sqrt{225}=15$, this Equation added to $x+y=\sqrt{625}$, gives $2x=\sqrt{625}+\sqrt{225}$, or $x=\frac{1}{2}\sqrt{625}+\frac{1}{2}\sqrt{225}=20$, therefore $y=\frac{1}{2}\sqrt{625}-\frac{1}{2}\sqrt{225}=5$.

QUESTION

QUESTION IV.

Given $\begin{cases} x^4+x^3y+x^2y^2+xy^3+y^4=211=a, \\ x^8+x^6y^2+x^4y^4+x^2y^6+y^8=11605=b. \end{cases}$

Divide the ſecond Equation by the firſt, add the Quotient $(x^4-x^3y+x^2y^2-xy^3+y^4=\frac{b}{a})$ to, and ſubtract it from the firſt Equation, and you will have $2x^4+2x^2y^2+2y^4=a+\frac{b}{a}$, or $x^4+x^2y^2+y^4=\frac{1}{2}a+\frac{b}{2a}$, and $2x^3y+2xy^3=a-\frac{b}{a}$, or $\overline{x^2+y^2}\times xy=\frac{1}{2}a-\frac{b}{2a}$.

Put $m=\frac{1}{2}a+\frac{b}{2a}=133$, $n=\frac{1}{2}a-\frac{b}{2a}=78$, $s=x^2+y^2$, and $p=xy$, then will our Equations become s^2-p^2 $(=\overline{x^2+y^2}|^2-x^2y^2=x^4+x^2y^2+y^4)=m$, and sp $(=\overline{x^2+y^2}\times xy)=n$, hence $p=\frac{n}{s}$, this Equation ſquared and added to $s^2-p^2=m$, gives $s^2=m+\frac{n^2}{s^2}$, or $s^4-ms^2=n^2$, whence we get $s=\sqrt{\frac{m}{2}+\sqrt{\frac{m^2}{4}+n^2}}=13$, therefore $p=\frac{n}{s}=6$: Now from the Equations $x^2+y^2=13=s$, and $xy=6=p$ (by proceeding as in the laſt Problem) you will find $x=\frac{1}{2}\sqrt{s+2p}|+\frac{1}{2}\sqrt{s-2p}|=3$, and $y=\frac{1}{2}\sqrt{s+2p}|-\frac{1}{2}\sqrt{s-2p}|=2$.

QUESTION V.

Given $\left.\begin{cases} x^2y+xy^2=180=a \\ x^2y^7+x^7y^2=1659600=b \end{cases}\right\}$ to find x and y.

Put $s=x+y$, and $p=xy$, then will $xy\times\overline{x+y}=ps=a$, or

$p=$

$p=\frac{a}{s}$, and $x^2y^2\times\overline{y^5+x^5}=p^2\times\overline{s^5-5s^3p+5sp^2}=b$ (See Page 157.)

This Equation by writing $\frac{a}{s}$, for p, becomes $\frac{a^2}{s^2}\times$ $\overline{s^5-5as^2+\frac{5a^2}{s}}=b$; hence $s^6-\overline{5a+\frac{b}{2a}}\times s^3=-5a^2$, and $s=\sqrt[3]{\frac{5a}{2}+\frac{b}{2a^2}+\sqrt{\overline{\frac{5a}{2}+\frac{b}{2a^2}}\,^2-5a^2}}=$ 9, and therefore $p=\frac{a}{s}=20$, hence $x+y=9$, and $xy=20$; four Times this Equation taken from the Square of $x+y=9$, leaves $x^2-2xy+y^2=1$, therefore $x-y=1$, consequently $x=5$, and $y=4$.

From the above Solution it will be easy to see how any Equation in this Form $x^n-nax^{n-2}+n.\frac{n-3}{2}.a^2x^{n-4}-n.\frac{n-4}{2}.\frac{n-5}{3}.a^3x^{n-6}+n.\frac{n-5}{2}.\frac{n-6}{3}.\frac{n-7}{4}.a^4x^{n-8}-$ &c. $=b$, may, by Substitution, be reduced to a Quadratic; for an Instance of this, assume $n=6$, then we shall have $x^6-6ax^4+9a^2x^2-2a^3=b$, the Sum of the Sixth Powers of two Numbers, by Problem 18, therefore putting $yz=p=a$, or $z=\frac{a}{y}$, and $y+z=x$, we likewise have $y^6+z^6=b$, or $y^6+\frac{a^6}{y^6}=b$, hence $y^{12}-by^6=-a^6$, here completing the Square, &c. $y^6=\frac{1}{2}b+\sqrt{\frac{1}{4}b^2-a^6}$, therefore $y=\sqrt[6]{\frac{1}{2}b+\sqrt{\frac{1}{4}b^2-a^6}}$, and consequently, x ($=y+z=y+\frac{a}{y}$) $=\sqrt[6]{\frac{1}{2}b+\sqrt{\frac{1}{4}b^2-a^6}}+\frac{a}{\sqrt[6]{\frac{1}{2}b+\sqrt{\frac{1}{4}b^2-a^6}}}$

Before

Before I proceed to the Solution of Cubic and other higher Equations, it may be proper to shew their Derivations.

These, like Quadratics, may be produced by the Multiplication of simple Equations.

Thus, suppose $x=a$, $x=b$, $x=c$, then will $x-a=0$, $x-b=0$, $x-c=0$; these three Equations multiplied together, give $\overline{x-a}\times\overline{x-b}\times\overline{x-c}=0$, or

$$x^3\left.\begin{matrix}-a\\-b\\-c\end{matrix}\right\}\times x^2\left.\begin{matrix}+ab\\+ac\\+bc\end{matrix}\right\}\times x-abc=0,\text{ a Cubic Equation,}$$

in which the three Roots are a, b, c; the Co-efficient of the first Term (x^3) is Unity, that of the second Term is $-a-b-c$ the Sum of all the Roots having their Signs changed; the Co-efficient of the third Term is $+ab+ac+bc$ the Sum of all the Products that can arise by multiplying any two of the Roots into one another.

The last Term is supposed to be a known Quantity, and is $-abc$, the Product of all the Roots multiplied continually together with their Signs changed.

In the Biquadratic Equation, $\overline{x-a}\times\overline{x-b}\times\overline{x-c}\times\overline{x-d}=0$, or

$$x^4\left\{\begin{matrix}-a\\-b\\-c\\-d\end{matrix}\right.\times x^3\left.\begin{matrix}+ab\\+ac\\+ad\\+bc\\+bd\\+cd\end{matrix}\right\}\times x^2\left.\begin{matrix}-abc\\-abd\\-acd\\-bcd\end{matrix}\right\}\times x+abcd=0.$$

The four Roots are a, b, c, d. The Co-efficient of the second Term is the Sum of the four Roots having contrary Signs.

The Co-efficient of the third Term is the Sum of all the Products that can be made, by multiplying any two of the Roots together.

The Co-efficient of the fourth Term is the Sum of all the Products that can arise by multiplying into one another any three of the Roots with their Signs changed.

 And

And the last Term is always the Product of all the Roots multiplied together with contrary Signs.

All this will ever hold true, even when some of the Roots are affirmative and the rest negative.

The Units in the Exponent of the highest Power of the unknown Quantity, are always equal in Number to the Roots in the Equation.

And when the Roots are all positive, the Signs will be alternately + and −, as above.

It is proposed to produce a Cubic Equation whose three Roots shall be 2, −3 and +5.

Let $x-2=0$, $x+3=0$, and $x-5=0$, then $\overline{x-2}\times\overline{x+3}\times\overline{x-5}=0$, or $x^3-4x^2-11x+30=0$, is the Equation required; for if instead of x you write 2, and for the Square and Cube of x you write the like Powers of 2, in the generated Equation $x^3-4x^2-11x+30=0$, it will become $8-16-22+30=0$; by writing -3 for x (in the same Manner) there arises $-27-36+33+30=0$; and by writing 5 for x, we have $x^3-4x^2-11x+30=125-100-55+30=0$.

In this Case it may be likewise observed, that the Co-efficient of the second Term $-4x^2$, is the Sum of the three proposed Roots, having their Signs changed, thus, $-2+3-5=-4$, the Co-efficient of the third Term $-11x$, is the Sum of all the Products that can arise by multiplying any two of the Roots together, thus, $2\times-3+2\times5+5\times-3=-11$; and the last Term 30, is the Product of all the Roots multiplied together, with contrary Signs, thus $3\times-5\times-2=+30$.

Again, let it be proposed to generate a biquadratic Equation whose four Roots shall be 3, 4, −5 and −7.

Here we have $\overline{x-3}\times\overline{x-4}\times\overline{x+5}\times\overline{x+7}=0$, which gives $x^4+5x^3-37x^2-101x+420=0$, for the Equation required.

And by proceeding on in this Manner, you may produce Equations which shall contain any Roots proposed.

In the Cubic Equation $x^3-4x^2-11x+30=0$, you may observe that there are two Changes of the Signs, one Change from the first Term to the Second, and one from the

the third to the fourth Term, which ſhew that there are two affirmative Roots; for there are as many poſitive Roots in any Equation as there are Changes of the Signs of the Terms from + to —, and from — to +; the two like Signs — and — which immediately follow one another from the ſecond to the third Term, denote one negative Root, for as often as two like Signs, either poſitive or negative, ſtand together in an Equation, ſo often there is a negative Root.

The whole of the Propoſition is general, where the Equations do not contain impoſſible Roots, or if the impoſſible Roots are allowed to be either affirmative or negative; and the Number of poſitive Roots being known, that of the negative ones (being the remaining Roots in the Equation) will, from thence be given.

Thus in the Equation $x^4-10x^3-7x^2+76x-60=0$, the Signs are + — — + —, and there are three Changes, from the firſt to the ſecond, from the third to the fourth, and from the fourth to the fifth Term, therefore there are three affirmative Roots, and the other Root is negative.

The negative Roots in any Equation may be turned into poſitive, and the poſitive into negative, by changing the Signs of the ſecond, fourth, ſixth, &c. Terms. Thus the Roots of the Biquadratic $x^4-10x^3-7x^2+76x-60=0$, which are 1, 2, -3 and 10, by changing the Signs of the ſecond and fourth Terms, the Equation becomes $x^4+10x^3-7x^2-76x-60=0$, whoſe Roots are -1, -2, $+3$ and -10.

Here you may obſerve, that the ſecond Term of this Equation is affirmative, which ſhews that the negative Roots, taken together, exceed the poſitive ones, which is very obvious here, and the Obſervation is general, extending to all Equations.

Of finding the Sums of the Powers of the Roots of an Equation.

Let $x^3-px^2+qx-r=0$ repreſent any Cubic Equation, then will p be the Sum of the Roots; q the Sum of the Products made by multiplying any two of them; r the

of all the three; and, if $-p, +q, -r, +s, -t, +u$, &c. be the Co-efficients of the 2*d*, 3*d*, 4*th*, 5*th*, 6*th*, &c. Terms of any Equation, then shall p be the Sum of all the Roots, q the Sum of the Products of any two, r the Sum of the Products of any three, s the Sum of the Products of any four, t the Sum of the Products of any five, u the Sum of the Products of any six, &c.

This being premised, it will be easy to find the Sum of the Squares, Cubes, Biquadrates, &c. of any determinate Number of Roots, for since p, the Co-efficient of the second Term, is the Sum of all the Roots having their Signs changed, it follows that the Sum of the Squares of any Number of Roots will be always equal to p^2-2q; for let B denote the Sum of the Squares, then because the Sum of the Roots is p, and the Square of the Sum of any Quantities is always equal to the Sum of their Squares, added to $(2q)$ double the Products that can be made by multiplying any two of them, therefore $p^2 = B+2q$, and consequently $B=p^2-2q$.

For Example, $\overline{a+b+c}|^2 = a^2+b^2+c^2+2ab+2ac+2bc = p^2 = B2q$; + likewise $\overline{a+b+c+d}|^2 = a^2+b^2+c^2+d^2+2\times\overline{ab+ac+ad+bc+bd+cd} = p^2 = B+2q$; therefore $B=p^2-2q$, and so for any other Number of Quantities. In general therefore, B the Sum of the Squares of the Roots may be always found by subtracting $2q$ from p^2, the Quantities p and q being always known, since they are Co-efficients in the proposed Equation.

If C be the Sum of the Cubes of the Roots in any Equation, then will $\overline{a^2+b^2+c^2-ab-ac-bc}\times\overline{a+b+c}$ $(=a^3+b^3+c^3-3abc) = \overline{B-q}\times p = C-3r$; hence $C=Bp-qp+3r$; here, by writing p^2-2q, for B, we have $C=p^3-3pq+3r$.

After the same Manner, if D be the Sum of the fourth Powers of the Roots, you will find $D=pC-qB+pr-4s$, from this Equation, by writing p^2-2q for B, and $p^3-3pq+3r$ for C, we get $D=p^4-4p^2q+4pr+2q^2-4s$; if E be the Sum of the fifth Powers, then will $E=pD-qC+rB-ps+5t=p^5-5p^3q+5p^2r+5pq^2-5qr-5ps+5t$.

And

And in this Manner the Sum of any Powers of the Roots may be obtained; the Progression of these Expressions of the Sum of the Powers being obvious.

The Sum of the Squares and Cubes of any Number of Roots is very readily found, and the Operations may be understood by Inspection; but in finding the Sums of higher Powers the Demonstrations are not so obvious; wherefore it may be acceptable to a curious Learner, to see the Sums of these Powers derived by a different Method.

In order to this, let $P = b+c+d$, &c. the Sum of all the Roots after the first (a), $Q = bc+bd+cd$, &c. $R = bcd + bce + bde$, &c. $S = bcde + bcef$, &c. and $T = bcdef$, &c. &c. then will

$$\begin{aligned} p &= a + P, \\ q &= Pa + Q, \\ r &= Qa + R, \\ s &= Ra + S, \\ t &= Sa + T, \text{ \&c.} \end{aligned}$$

From the Square of the first of these Equations take twice the second, and you will have $(B=)\ p^2 - 2q = a^2 + P^2 - 2Q$, the Sum of the Squares, which multiply by the Equation $p = a + P$, and from the Product subtract that of the Equations $p = a + P$ and $q = Pa + Q$ multiplied together, to the Remainder $pB - pq = a^3 - 3Qa + P^3 - 3PQ$, add three Times the Equation $r = Qa + R$, and there will arise $(C=)\ pB - pq + 3r = a^3 + P^3 - 3PQ + 3R$, the Sum of the Cubes, which multiplied by $p = a + P$, and from the Product, that of the Equations $q = Pa + Q$ and $B = a^2 + P^2 - 2Q$, being subtracted, there remains $pC - qB = a^4 - Qa^2 - PQa + 3Ra + P^4 - 4P^2Q + 3PR + 2Q^2$, to this Equation, in order to destroy all the Terms in which a is involved except the first (a^4), add $pr = Qa^2 + PQa + Ra + PR$, and from the Sum subtracting four Times $s = Ra + S$, we get $(D=)\ pC - qB + pr - 4s = a^4 + P^4 - 4P^2Q + 4PR + 2Q^2 - 4S$, the Sum of the Biquadrates; which, being likewise multiplied by $p = a + P$, and from the Product that of the Equations $q = Pa + Q$ and $C = a^3 + P^3 - 3PQ + 3R$, subtracted, leaves $pD - qC = a^5 + P^5 - 5P^3Q - P^2Qa + 4P^2R + 5PQ^2 + PRa - Qa^3 + 2Q^2a - 3QR - 4Sa - 4PS$, to this Equation add $rB = Qa^3 - 2Q^2a + P^2Qa + Ra^2 + P^2R - 2Q$

R,

R, together with five Times the Equation $t=Sa+T$, and from the Sum, take $ps=Ra^2+PRa+Sa+PS$, and there will remain $pD-qC+rB-ps+5t=a^5+P^5-5P^3Q+5P^2R+5PQ^2-5QR-5PS+5T=E$, the Sum of the fifth Powers, as before; hence the Law of Continuation is also manifest, the Sum (F) of the sixth Powers being $pF-qD+rC-sB+pt-6u$.

When all the Roots of an Equation are negative, then $\overline{x+a}\times\overline{x+b}\times\overline{x+c}\times\overline{x+d}$, &c. $=0$, will express the Equation to be produced, all whose Terms will evidently be positive; so that when all the Roots of an Equation are negative, it is plain there will be no Changes in the Signs of the Terms in that Equation.

Of Commensurable Quantities.

QUANTITIES are said to be commensurable when they are to each other as Number to Number, that is, when their Proportion may be expressed in whole Numbers, having one common Divisor that will measure them all.

For Instance, let a and b be two commensurable Quantities, then their Proportion, be it what it will, may be expressed in whole Numbers: for let c be their common Measure without Remainder, and let it measure a just three Times, and b four Times; then will $3c=a$, and $4c=b$; therefore $a:b::(3c:4c::)\ 3:4$, which was to be demonstrated.

Hence (*vice versa*) all Quantities that are to one another as Number to Number, are commensurable; therefore all whole Numbers are commensurable, since Unity is a common Measure to them all.

All Fractions are likewise commensurable, as $\frac{a}{b}$ and $\frac{c}{d}$; for if they are reduced to a common Denominator,

they

they will become $\frac{ad}{bd}$ and $\frac{bc}{bd}$ and $\frac{1}{bd}$ will measure them both.

Incommensurable Quantities are such whose Proportion cannot be expressed in whole Numbers or finite Fractions, and therefore have no common Measure, but their Relation must be expressed by a Surd Quantity or Infinite Series; thus the Ratio of 1 to $\sqrt{2}$, cannot be expressed in Numbers.

The Resolution of Equations, whose Roots are commensurate.

IT has been demonstrated, that the last Term of any Equation is the Product of all its Roots; from which it follows, that the Roots of an Equation, when commensurable Quantities will be found among the Divisors of the last Term, and hence we have for the Resolution of Equations this

RULE.

Bring all the Terms to one Side of the Equation, find all the Divisors of the last Term, substitute them successively for the unknown Quantity in the Equation; so shall that Divisor which (substituted in this Manner) gives the Result $=0$, be a Root of the Equation.

For Example, suppose this Equation is to be solved.

$$\left.\begin{array}{l} x^3-3ax^2+2a^2x-2a^2b \\ \quad -bx^2\;+3abx \end{array}\right\} =0,$$

where the last Term is $2a^2b$, whose simple literal Divisors are a, b, $2a$, $2b$, each of which may be taken either positively or negatively; but as here the Signs of the Terms are alternately $+$ and $-$, therefore we need only take them positively. Suppose $x=a$ the first of the Divisors, and writing a for x, the Equation becomes

a^3-

$$\left.\begin{matrix} a^3-3a^3+2a^3-2a^2b \\ -a^2b+3a^2b \end{matrix}\right\} \text{ or } 3a^3-3a^3+3a^2b-3a^2b=0$$

Here all the Terms destroy one another, therefore a is one Root of the Equation, and by writing b for x in the proposed Equation, it becomes

$$\left.\begin{matrix} b^3-3ab^2+2a^2b-2a^2b \\ -b^3+3ab^2 \end{matrix}\right\} = 0.$$

Where the Terms also destroy each other, which shews that b is another Root; these two Roots being obtained, the third may be found by dividing the last Term, having its Sign changed, by their Product, thus $\frac{2a^2b}{ab}=2a$ is the third Root, as you will find by substituting $2a$ for x, in the given Equation.

Let Equation $x^3-4x^2-7x+10=0$, be proposed; then the Divisors of (10) the last Term being $+1, -1, +2-2, +5, -5, +10, -10$, let these Numbers be successively substituted instead of x, and we shall have

$1-4-7+10=0$, therefore 1 is a Root;
$-1-4+7+10=12$, therefore -1 is no Root;
$8-16-14+10=-12$, therefore 2 is no Root;
$-8-16+14+10=0$, therefore -2 is another Root;
$125-100-35+10=0$, therefore 5 is the third Root.

This third Root might have been found by dividing the last Term 10, (having its Sign changed) by -2, the Product of the other two Roots.

Sometimes the Divisors of the last Term are very numerous; in which Case, to avoid Trouble, it will be convenient to transform the Equation to another, wherein the Divisors are fewer; this may be done by increasing, or diminishing the Roots by some known Quantity; and having discovered one of the Roots, if it is a Cubic Equation, divide it by the simple Equation, which you are to deduce from the Root already found; if it is a Biquadratic that is to be transformed, and you can find two of its Roots, divide it by the Product of the simple Equations deduced from those Roots, and so on for higher Equations, then you may readily find the remaining Roots (be they what

they

they will) by solving the Quadratic Equations thence arising.

For Example, Let there be given $x^3-19x^2+118x-240=0$.

Then, by writing $y+4$ for x, in the given Equation, it will be transformed to $y^3-7y^2+14y-8=0$; here the Divisors of 8, the last Term are very few; and it appears by Inspection that one of the Values of y is Unity; and since $y=1$, therefore one of the Values of x is 5, (for $x=y+4=5$, hence $x-5=0$, and dividing the given Equation $x^3-19x^2+118x-240=0$ by $x-5$, the Quadratic Quotient is $x^2-14x+48=0$, hence $x^2-14x=-48$, this Equation solved, gives $x=7\pm\sqrt{1}=8$, or 6; so that the three Roots of the original Equation are 5, 6, and 8.

Let there be proposed the Biquadratic $x^4-4x^3-8x+32=0$, and in order to change it to another whose last Term admits of fewer Divisors, write $y+1$ for x, and it will become $y^4-6y^2+16y+21=0$; where the Divisors of the last Term are $1, -1, 3, -3, 7, -7, 21$, and -21, which being substituted successively for y, I find the Divisors 1 and 3 to succeed, and since two Values of y are 1 and 3, it follows that two of the Values of x are 2 and 4; therefore $x-2=0$, likewise $x-4=0$, and consequently $\overline{x-4}\times\overline{x-2}=0$, that is, $x^2-6x+8=0$, and dividing the proposed biquadratic Equation by x^2-6x+8, it becomes $x^2+2x+4=0$, a Quadratic, which, being solved, gives $x=-1\pm\sqrt{-3}$; therefore the four Roots of the Biquadratic are $2, 4, -1+\sqrt{-3}$ and $-1-\sqrt{-3}$. But the last two of them are evidently impossible.

There is another Method by which the Divisors that will reduce Equations to lower Dimensions may be found; and for the better understanding of this, transform the Cubic Equation $x^3-px^2+qx-r=0$, into two others whose Roots shall in each differ from those of the proposed Equation by Unity; thus by writing $y+1$, and $y-1$, successively for x, in the Cubic Equation, it will be transformed to

y^3+

$$\left.\begin{array}{r} y^3+3y^2+3y+1 \\ -py^2-2py-p \\ +qy+q \\ -r \end{array}\right\} = 0 \text{ and to}$$

$$\left.\begin{array}{r} y^3-3y^2+3y-1 \\ -py^2+2py-p \\ qy-q \\ -r \end{array}\right\} = 0.$$

The Values of the y's are some Divisors of $+1-p+q-r$, and of $-1-p-q-r$, the last Terms of the Equations derived by Transformation; and if you write $+1$, 0, -1, successively for x in the given Equation, it will become $+1-p+q-r=0$, * * * $-r=0$, and $-1-p-q-r=0$.

The Terms which constitute the first and last of these three Equations, are respectively the same as those composing the last Terms of the two Equations in which y is involved; and the Values of x are some Divisors of r, the Term left by writing 0 for x: But the Terms $+1$, 0, -1, are in Arithmetical Progression, decreasing by the common Difference Unity, and the Values of the y's, namely $x-1$, x, $x+1$, are in that Progression increasing by Unity. And it is obvious the same Reasoning may be extended to any Equation of whatever Degree; so that this gives a general Rule for finding the commensurable Roots of Equations, which is thus:

Substitute in place of the unknown Quantity successively the Terms of the Progression 1, 0, -1, &c. and find all the Divisors of the Sums that result; then take out all the Arithmetical Progressions you can find among the Divisors, whose common Difference is Unity; and the Values of x will be among the Divisors arising from the Substitution of $x=0$ that belong to these Progressions. The Values of x will be affirmative when the Arithmetical Progression encreases, but negative when it decreases.

EXAMPLE

EXAMPLE

Let it be required to find one Root of the Equation $x^3 - x^2 - 10x + 6 = 0$. Vide the Operation.

Suppofition	Refults.	Divifors.	Arith. Prog. Dec.
$x = 1$	$x^3 - x^2 - 10x + 6 = -4$	1.2.4.	4
$x = 0$	$x^3 - x^2 - 10x + 6 = +6$	1.2.3.6	3 gives $x = -3$.
$x = -1$	$x^3 - x^2 - 10x + 6 = +14$	1.2.7.14	2

Here you fee the Suppofitions of $x = 1$, $x = 0$, $x = -1$, give the Refults of the Quantity $x^3 - x^2 - 10x + 6$ equal to -4, 6, 14; among whofe Divifors I find only one Arithmetical Progreffion 4, 3, 2; the Term of which oppofite to the Suppofition of $x = 0$, being 3, and the Series decreafing, I try if the propofed Equation be divifible by $x - 3$, without Remainder, and find it to fucceed, the Quotient being $x^2 - 4x + 2 = 0$; therefore -3 is one of the Roots; and from the Equation $x^2 - 4x = -2$, you will find the other two Roots to be $2 \pm \sqrt{2}$.

If it is required to find the Roots of the Biquadratic Equation $x^4 - 5x^3 - 7x^2 + 8x - 12 = 0$, the Work will ftand thus:

Suppofition.	Refults.	Divifors.	Progreffions.
$x = 1$	-15	1, 3. 5. 15.	1 3 5 5
$x = 0$	-12	1, 2. 3. 4. 6. 12.	2 2 4 6
$x = -1$	-21	1, 3. 7. 21.	3 1 3 7

In thefe four Progreffions, the Terms correfponding to the Suppofition of $x = 0$, being 2, 2, 4, 6, and the firft Progreffion 1, 2, 3, being increafing, I try if $+2$ fubftituted for x in the Biquadratic will give the Refult equal to nothing, which not fucceeding, I therefore try -2 becaufe the fecond Progreffion 3, 2, 1, is decreafing, and find it to fucceed; in like Manner I try -4 and $+6$, and find that $+6$ fubftituted for x in the Equation gives the Refult $= 0$; therefore -2 and $+6$ are two of its Roots, therefore $x + 2 = 0$, and $x - 6 = 0$, and confequently $\overline{x+2} \times \overline{x-6} = 0$, that is, $x^2 - 4x - 12 = 0$, and dividing the propofed

proposed Equation by $x^2-4x-12$, the Quadratic Quotient will be $x^2-x+1=0$; hence $x^2-x=-1$, this Equation solved, gives $\frac{1}{2}\pm\sqrt{-\frac{3}{4}}$, for the other two Roots required.

To transform any Equation to another that shall want its second Term.

RULE.

Divide the Co-efficient of the second Term by the Units in the Exponent of the highest Power of the unknown Quantity; to the Quotient having its Sign changed, add a new unknown Quantity, and write their Sum for the original unknown Quantity and its Powers in the proposed Equation, and there will arise an Equation wanting its second Term.

EXAMPLE.

Let it be required to exterminate the second Term out of this Equation $y^3-15y^2+81y-243=0$.

Here, dividing the Co-efficient of the second Term $-15y$ by 3 (the Exponent of y^3) changing the Sign of the Quotient -5, and writing $x+5$ for y in the proposed Equation, there will arise.

$$\left.\begin{array}{rl} +y^3 = & x^3+15x^2+75x+125 \\ -15y^2 = & -15x^2-150x-375 \\ +81y = & +81x+405 \\ -243 = & -243 \end{array}\right\} = 0;$$

Whence by Addition we have ($y^3-15y^2+81y-243=$) $x^3+6x-88=0$, an Equation wanting the second Term; and since any Equation may be reduced to this Form;

I shall therefore proceed (by a general Method) to the Solution of Cubic Equations which want their second Terms.

EXAMPLE

EXAMPLE I.

Let there be given $x^3+6x=88$, to find x.

Put $c=6$, $s=88$, $3mn=c$, or $n=\frac{c}{3m}$, and $m-n=x$:

Then by writing $m-n$ for x, the given Equation will become $m^3-3m^2n+3mn^2-n^3+cm-cn=s$; to this, add the Equation $3mn=c$, multiplied by $m-n$, and you will have $m^3-n^3+cm-cn=s+cm-cn$, or $m^3-n^3=s$, and by writing $\frac{c}{3m}$ for n, there arises $m^3-\frac{c^3}{27m^3}=s$, hence $m^6-sm^3=\frac{c^3}{27}$, here by completing the Square, &c. we get $m^3=\frac{1}{2}s+\sqrt{\frac{c^3}{27}+\frac{s^2}{4}}$, consequently $m=\sqrt[3]{\frac{1}{2}s+\sqrt{\frac{c^3}{27}+\frac{s^2}{4}}}$; and since $n\left(=\frac{c}{3m}\right)=\frac{\frac{1}{3}c}{m}$ therefore $n=\frac{\frac{1}{3}c}{\sqrt[3]{\frac{1}{2}s+\sqrt{\frac{c^3}{27}+\frac{s^2}{4}}}}$ and, consequently $x\ \left(=m-n=m-\frac{\frac{1}{3}c}{m}\right)=\sqrt[3]{\frac{1}{2}s+\sqrt{\frac{c^3}{27}+\frac{s^2}{4}}}-\frac{\frac{1}{3}c}{\sqrt[3]{\frac{1}{2}s+\sqrt{\frac{c^3}{27}+\frac{s^2}{4}}}}=\sqrt[3]{44+\sqrt{1944}}-\frac{2}{\sqrt[3]{44+\sqrt{1944}}}=\sqrt[3]{44+44{,}090815}-\frac{2}{\sqrt[3]{44+44{,}090815}}$ $=4{,}449-0{,}449=4$; and y in the original Equation ($y^3-15y^2+81y-243=0$), being $=x+5=9$.

O EXAMPLE

EXAMPLE II.

Given $x^3-21x=90$, to find x.

Here we have $c=21$, $s=90$, $3mn=c$, or $n=\frac{\frac{1}{3}c}{m}$, and because the Term $-21x$ is negative, therefore write $m+n$ for x, in the given Equation, and it will be transformed to $m^3+3m^2n+3mn^2+n^3-cm-cn=s$, from this subtracting the Equation $3mn=c$, multiplied by $m+n$, there remains $m^3+n^3-cm-cn=s-cm-cn$, or $m^3+n^3=s$, and by writing $\frac{\frac{1}{3}c}{m}$, or $\frac{c}{3m}$ for n, we have $m^3+\frac{c^3}{27m^3}=s$;

hence $m^6-sm^3=-\frac{c^3}{27}$, and $m=\sqrt[3]{\frac{1}{2}s+\sqrt{\frac{s^2}{4}-\frac{c^3}{27}}}$,

and because $n=\frac{\frac{1}{3}c}{m}$, therefore $x\left(=m+\frac{\frac{1}{3}c}{m}\right)=$

$$\sqrt[3]{\tfrac{1}{2}s+\sqrt{\frac{s^2}{4}-\frac{c^3}{27}}}+\frac{\frac{1}{3}c}{\sqrt[3]{\tfrac{1}{2}s+\sqrt{\frac{s^2}{4}-\frac{c^3}{27}}}}$$

$$=\sqrt[3]{45+\sqrt{\frac{8100}{4}-\frac{9261}{27}}}+\frac{7}{\sqrt[3]{45+\sqrt{\frac{8100}{4}-\frac{9261}{27}}}}$$

$=\sqrt[3]{86,012193}+\frac{7}{\sqrt[3]{86,012193}}=4,414213+1,585787=6.$

EXAMPLE III.

Suppose $x^3-12x=16$, Quere, the Value of x.

By writing in the last Theorem, 12 for c and 16 for s, we

have $x=\sqrt[3]{\frac{1}{2}s+\sqrt{\frac{s^2}{4}-\frac{c^3}{27}}}+\frac{\frac{1}{3}c}{\sqrt[3]{\frac{1}{2}s+\sqrt{\frac{s^2}{4}-\frac{c^3}{27}}}}$

$$= \sqrt[3]{8+\sqrt{\frac{256}{4}-\frac{1728}{27}}} + \sqrt[3]{8+\sqrt{\frac{256}{4}-\frac{1728}{27}}}:$$

But $\frac{1728}{27}$ is equal to $\frac{256}{4}$, consequently $\frac{256}{4} - \frac{1728}{27} = 0$, and therefore $x = \sqrt[3]{8} + \sqrt[3]{8} = 2 + 2 = 4$.

Hence you may observe that this Method will fail when $\frac{c^3}{27}$ is negative, and, at the same Time, greater than $\frac{s^2}{4}$; for then $\frac{s^2}{4} - \frac{c^3}{27}$ will be a negative Quantity, and its square Root will therefore be impossible: But when such Equations occur, they may be easily solved by converging Series, as shall be shewn hereafter.

Note. The Quantity $\frac{s^2}{4} - \frac{c^3}{27}$ will be negative, unless two Roots of the proposed Equation are impossible; except when two of them are equal, as in the last Example, in which the other two Roots are -2 and -2.

Hence you may likewise observe, that the positive Root 4, is equal to the Sum of the two negative Roots; and in any Equation where the second Term is wanting, the Sum of the negative Roots is always equal to the Sum of the affirmative ones.

It may not be improper to remark here, that a Cubic Equation, having all its Terms, if the Co-efficient of its second Term be divided by a Number which will produce 3 in the Quotient, and if the Co-efficient of the third Term be divided by the Square of the same Number, and the Quotient be likewise 3, it may be solved by writing the Cube of $\frac{1}{3}$ of the Co-efficient of the second Term, with its proper Sign on both Sides of the Equation, and then extracting the Cube Root, as in the following Examples.

O 2 EXAMPLE

EXAMPLE I.

Given $x^3+6x^2+12x=992=s$, to find x.

Here it is obvious that this Equation is one of the Claſs above deſcribed; for putting $a=2$, we have $6=3a$, and $12=3a^2$, therefore the given Equation may be expreſſed thus, $x^3+3ax^2+3a^2x=s$; here a being $\frac{1}{3}$ of the Co-efficient of the ſecond Term, I add its Cube a^3, to both Sides of the Equation, and it becomes $x^3+3ax^2+3a^2x+a^3=a^3+s$, hence $x+a=\sqrt[3]{a^3+s}$, and $x=\sqrt[3]{a^3+s}-a=8$.

EXAMPLE II.

Suppoſe $x^3-12x^2+48x=72$.

Here, by completing the Cube, we have $x^3-12x^2+48x-64=72-64=8$, hence $x-4=\sqrt[3]{8}$, therefore $x=\sqrt[3]{8}+4=6$.

EXAMPLE III.

Let $x^3-15x^2+75x=121{,}625$.

This Equation ſolved, gives $x=5+\sqrt[3]{-3{,}375}=5-1{,}5=3{,}5$.

SCHOLIUM.

If the Signs of the Terms of a Cubic Equation be as in either of theſe Examples, and the Co-efficients to one another in the ſame Proportion, then it may be always ſolved by completing the Cube, &c.

The

The Resolution of Biquadratics, performed by assuming the Difference between two complete Squares, equal to the proposed Biquadratic Equation, and thereby destroying its two first Terms.

EXAMPLE I.

Given $x^4+4x^3-47x^2+6x+216=0$, to find x.

By writing a, b, c, and d respectively for 4, 47, 6 and 216, in the given Equation, it becomes $x^4+ax^3-bx^2+cx+d=0$: Assume $\overline{x^2+\frac{1}{2}ax+A}|^2-\overline{Bx+C}|^2=x^4+ax^3-bx^2+cx+d$; here A, B, and C represent unknown Quantities to be determined, and the Quantities $x^2+\frac{1}{2}ax+A$, and $Bx+C$ being each actually involved to the second Power, the two first Terms x^4+ax^3, on each Side of the Equation, will destroy one another; and so we shall have $\overline{2A+\frac{1}{4}a^2-B^2}\times x^2+\overline{aA-2BC}\times x+A^2-C^2=-bx^2+cx+d$: Here, by making the Co-efficients of x^2 and of x on one side of this Equation, respectively equal to those of the same Powers of x on the other Side, we have $2A+\frac{1}{4}a^2-B^2=-b$, and $aA-2BC=c$; or $B=\sqrt{2A+\frac{1}{4}a^2+b}|$, and $B=\frac{aA-c}{2C}$, hence $C=\frac{aA-c}{2B}$; and by equating the rest of the Terms we have $A^2-C^2=d$; or $C=\sqrt{A^2-d}|$; write $\sqrt{A^2-d}|$ for C in the foregoing Equation $B=\frac{aA-c}{2C}$, and you will have $B=\frac{aA-c}{2\sqrt{A^2-d}|}$; therefore $\sqrt{2A+\frac{1}{4}a^2+b}|=\frac{aA-c}{2\sqrt{A^2-d}|}$ here, by squaring both Sides, &c. we have $8A^3+a^2A^2+4bA^2-8dA-a^2d-4bd=a^2A^2-2acA+c^2$, or $A^3+\frac{1}{2}bA^2+\overline{\frac{1}{4}ac-d}\times A-\frac{1}{2}bd-\frac{1}{8}a^2d-\frac{1}{8}c^2=0$: In Numbers $A^3+23{,}5\,A^2-210\,A-5512{,}5=0$.

Here it is obvious by bare Inspection, that A is greater than 10, and less than 20; now it is plain that when one Number measures another, it must necessarily measure its double, triple, &c.

In this Example the double of the last Term 5512,5 is 11025 an odd Number, and since an even Number cannot

measure an odd one, it is needless therefore to try an even Number for a Divisor, and there are only five odd Integers, namely 11, 13, 15, 17 and 19 between these two Limits 10 and 20, and but one of them will divide 11025 without a Remainder, and that is the Number 15, this, by Trial, I find to be a Root of the Equation $A^3 + 23,5A^2 - 210A - 5512,5 = 0$; therefore $A = 15$.

Hence $B = \sqrt{\overline{2A + \frac{1}{4}a^2 + b}} = \sqrt{\overline{81}} = 9$; and $C = \frac{aA - c}{2B} = \frac{60 - 6}{18} = 3$: And because the proposed Biquadratic is equal to (0) nothing, therefore $\overline{x^2 + \frac{1}{2}ax + A}|^2 - \overline{Bx + C}|^2 = 0$, and consequently $\overline{x^2 + \frac{1}{2}ax + A}|^2 = \overline{Bx + C}|^2$ hence, by extracting the square Root we have $x^2 + \frac{1}{2}ax + A = \pm Bx + C$, or $x^2 + \frac{1}{2}ax \pm Bx = \pm C - A$; this Equation solved, gives $x = \pm \frac{1}{2}B - \frac{1}{4}a \pm \sqrt{\overline{\frac{1}{16}a^2 \mp \frac{1}{4}aB + \frac{1}{4}B^2 \pm C - A}} = \pm \frac{9}{2} - 1 \mp \sqrt{\overline{1 \mp 9 + \frac{81}{4} \pm 3 - 15}}$. Therefore the four Roots of the given Equation are

$$x = \begin{cases} +\frac{9}{2} - 1 + \sqrt{\overline{1 - 9 + \frac{81}{4} + 3 - 15}} = 3,5 + ,5 = 4, \\ +\frac{9}{2} - 1 - \sqrt{\overline{1 - 9 + \frac{81}{4} + 3 - 15}} = 3,5 - ,5 = 3; \\ -\frac{9}{2} - 1 + \sqrt{\overline{1 + 9 + \frac{81}{4} - 3 - 15}} = -5,5 + 3,5 = -2, \text{and} \\ -\frac{9}{2} - 1 - \sqrt{\overline{1 + 9 + \frac{81}{4} - 3 - 15}} = -5,5 - 3,5 = -9. \end{cases}$$

EXAMPLE II.

Let it be required to find all the Roots of this Equation $x^4 - 10x^3 + 8x^2 + 106x - 105 = 0$.

Here $a = 10$, $b = 8$, $c = 106$, $d = 105$; and because the second Term $-10x^3$ is negative: Put $\overline{x^2 - \frac{1}{2}ax + A}|^2 - \overline{Bx + C}|^2 = x^4 - ax^3 + bx^2 + cx - d = 0$; then will $2Ax^2 + \frac{1}{4}a^2x^2 - aAx + A^2 - B^2x^2 - 2BCx - C^2 = bx^2 + cx - d$; hence, by equating the homologous Terms, as before, we have $B^2 = 2A + \frac{1}{4}a^2 - b$, $-2BC = aA + c$, and $C^2 = A^2 + d$.

From these three Equations (by proceeding as in the first Example) we get $A^3 - \frac{1}{2}bA^2 + \overline{d - \frac{1}{4}ac} \times A + \frac{a^2d - 4bd - c^2}{8} = 0$, in Numbers $A^3 - 4A^2 - 160A - 512 = 0$; here, by trying the Divisors of the last Term (512) you

you will find $A = 16$, then $B = \sqrt{2A + \frac{1}{4}a^2 - b} = \sqrt{49} = 7$, and from the Equation $-2BC = aA + c$, we have $C = \frac{-aA - c}{2B} = \frac{-266}{14} = -19$; and because $\overline{x^2 - \frac{1}{2}ax + A}|^2 - \overline{Bx + C}|^2 = 0$, therefore $\overline{x^2 - \frac{1}{2}ax + A}|^2 = \overline{Bx + C}|^2$, or $x^2 - \frac{1}{2}ax + A = \pm Bx + C$; therefore $x^2 - \frac{1}{2}ax \pm Bx = \pm C - A$; hence $x = \pm \frac{1}{2}B + \frac{1}{4}a \pm \sqrt{\frac{1}{16}a^2 \pm \frac{1}{4}aB + \frac{1}{4}B^2 \pm C - A} = \pm 3,5 + 2,5 \mp \sqrt{6,25 \mp 17,5 + 12,25 \pm 19 - 16.}$

Here the four Roots of the proposed Equation are,

$$-3,5 + 2,5 + \sqrt{2,5 - 17,5 + 19} = 1,$$
$$-3,5 + 2,5 - \sqrt{2,5 - 17,5 + 19} = -3;$$
$$+3,5 + 2,5 + \sqrt{2,5 + 17,5 - 19} = 7, \text{ and}$$
$$+3,5 + 2,5 - \sqrt{2,5 + 17,5 - 19} = 5.$$

EXAMPLE III.

Given $x^4 + 12x - 17 = 0$, to find the several Values of x.

Here the last Term -17 being negative, the Example belongs to Case II, and because the second and third Terms are wanting, we have $a = 0$, $b = 0$; $c = 12$, and $d = 17$; and therefore by rejecting all those Terms in which a and b are involved in the Cubic Equation $A^3 - \frac{1}{2}bA^2 + \overline{d - \frac{1}{4}ac} \times A + \frac{a^2d - 4bd - c^2}{8} = 0$, it becomes $A^3 + dA - \frac{c^2}{8} = A^3 + 17A - 18 = 0$:

Here it is obvious that $A = 1$; hence $B = \sqrt{2A + \frac{1}{4}a^2 - b} = \sqrt{2A} = \sqrt{2}$, and $C \left(= \frac{-aA - c}{2B}\right) = \frac{-c}{2B} = \frac{-12}{2\sqrt{2}} = -3\sqrt{2}$. Now by writing 1, $\sqrt{2}$ and $-3\sqrt{2}$ respectively for A, B, and C, in the second general Theorem, namely, in $x = \pm \frac{1}{2}B + \frac{1}{4}a \pm \sqrt{\frac{1}{16}a^2 \mp \frac{1}{4}aB + \frac{1}{4}B^2 \pm C - A}$, (excluding the Terms involving a) you will have $x = \pm \frac{1}{2}\sqrt{2} \pm \sqrt{\frac{1}{2} \mp 3\sqrt{2} - 1}$

 =

$= \pm\frac{1}{2}\sqrt{2} \mp \sqrt{\pm 3\sqrt{2} - \frac{1}{2}}$. Therefore the four Roots of the given Equation are $\frac{1}{2}\sqrt{2} + \sqrt{-3\sqrt{2} - \frac{1}{2}}$, $\frac{1}{2}\sqrt{2} - \sqrt{-3\sqrt{2} - \frac{1}{2}}$, $-\frac{1}{2}\sqrt{2} + \sqrt{3\sqrt{2} - \frac{1}{2}}$, and $-\frac{1}{2}\sqrt{2} - \sqrt{3\sqrt{2} - \frac{1}{2}}$; whereof the two first are impossible.

EXAMPLE IV.

Given $x^4 + 8x^3 + 18x^2 - 20x - 67 = 0$, to find x.

By assuming $\overline{x^2 + \frac{1}{2}ax + A}|^2 - \overline{Bx + C}|^2$ equal to the given Equation; and equating the homologous Terms, you will find $B^2 = 2A + \frac{1}{4}a^2 - b$, $2BC = aA + c$, and $C^2 = A^2 + d$. Here a, b, c, and d, represent the known Numbers 8, 18, 20 and 67 respectively; hence $A^3 - \frac{1}{2}bA^2 + \overline{d - \frac{1}{4}ac}$

$$\times A + \frac{a^2d - 4bd - c^2}{8} = A^3 - 9A^2 + 27A - 117 = 0, \text{ or}$$

$A^3 - 9A^2 + 27A = 117$; here, by completing the Cube, &c. we get $A = 3 + \sqrt[3]{90}$.

Put $m = 3 + \sqrt[3]{90}$, $n\ (=B) = \sqrt{2A + \frac{1}{4}a^2 - b}$, and $r\ (=C) = \dfrac{aA + c}{2B}$: Then, by writing m, n, and r respectively for A, B and C, in the first general Theorem we have $x = \pm\frac{1}{2}n - \frac{1}{4}a \pm \sqrt{\frac{1}{16}a^2 \pm \frac{1}{4}an + \frac{1}{4}n^2 \pm r - m}$, the four Roots of the proposed Equation.

By assuming all the Terms of a Biquadratic Affirmative, and proceeding as in the first of these Examples, you will find the Cubic Equation thence arising to be $A^3 - \frac{1}{2}bA^2 + \overline{\frac{1}{4}ac - d} \times A + \frac{1}{2}bd - \frac{1}{8}a^2d - \frac{1}{8}c^2 = 0$.

In the Use of this Theorem, a, b, c, and d, are to be taken equal to either positive or negative Numbers, so as to correspond respectively with the Signs of the Terms of the proposed Biquadratic to be solved, as shall be shewn further on.

Having

Having given a general Solution of Biquadratic Equations by Means of Cubic ones, I shall now point out, and illustrate, a few Cases in which the Resolution of Biquadratics can be performed by Quadratics only.

These may be discovered from the preceding Equations collected, namely,

$$B^2 = 2A + \tfrac{1}{4}a^2 - b,$$
$$2BC = aA - c,$$
$$\text{And } C^2 = A^2 - d.$$

Wherein, if A be supposed $=0$, it is plain that $C^2 = -d$, or $C = \sqrt{-d}$, also $2BC = -c$, or $B = \frac{-c}{2C} = \frac{-c}{2\sqrt{-d}}$ and $B^2 = \frac{1}{4}a^2 - b$, or $B = \sqrt{\frac{1}{4}a^2 - b}$, therefore $\sqrt{\frac{1}{4}a^2 - b} = \frac{-c}{2\sqrt{-d}}$, hence we get $\frac{1}{4}a^2 - b = \frac{c^2}{-4d}$, or $4bd - ad = c^2$, and $d = \frac{c^2}{4b - a^2}$: Hence you are to observe, that a Biquadratic Equation may be solved by a Quadratic, when its last Term is equal to the Square of the Co-efficient of the fourth Term divided by four Times that of the third Term, minus the Square of that of the second: for

EXAMPLE.

Let there be given $x^4 - 10x^3 + 16x^2 + 108x - 324 = 0$.

Here $a = -10$, $b = 16$, $c = 108$; hence $\frac{c^2}{4b - a^2} = \frac{11664}{64 - 100} = \frac{11664}{-36} = -324 =$ the last Term (d), agreeable to the Observation; and because $d = -324$, therefore $-d = 324$, consequently $C = \sqrt{-d} = \sqrt{324} = 18$, and $B = \frac{-c}{2C} = \frac{-108}{36} = -3$: And by writing 10, 3 and 18 respectively for a, B and C, in the second general

neral Equation $x^2 - \frac{1}{2}ax \pm Bx = \pm C - A$, (A being $=0$) we have $x^2 - 5x \pm 3x = \pm 18$; hence by completing the Square, &c. we get $x = \pm\frac{3}{2} + \frac{5}{2} \pm \sqrt{\frac{25}{4} \pm \frac{30}{4} + \frac{9}{4} \pm 18}$ $= \pm\frac{3}{2} + \frac{5}{2} \mp \sqrt{8,5 + 7,5 \pm 18}$.

Therefore $1 + \sqrt{19}$, $1 - \sqrt{19}$, $4 + \sqrt{-2}$, and $4 - \sqrt{-2}$, are the four Roots of the given Equation, whereof the two last are impossible.

Note. The Values of x may be more readily obtained, by substituting those of A, B, and C in the Theorems derived respectively from the general Equations. But if the second Term of the given Equation be wanting, then the Values of A, B, and C may be written in either of the two general Theorems.

Suppose $x^4 - 25x^2 + 60x - 36 = 0$, Quere, the Values of x.

Here $a = 0$, $b = -25$, $c = 60$; hence $d \left(= \frac{c^2}{4b - a^2}\right)$ $= \frac{c^2}{4b} = \frac{3600}{-100} = -36$, or $-d = 36$, therefore C $=$ $\sqrt{-d} = \sqrt{36} = 6$, and $B = \frac{-c}{2C} = \frac{-60}{12} = -5$.

Now by writing 5 for B, and 6 for C, in either of the general Theorems, you will have $x = \pm\frac{5}{2} \pm \sqrt{6,25 \mp 6}$, therefore the four Roots of the proposed Equation are $+\frac{5}{2} + \sqrt{,25}$, $+\frac{5}{2} - \sqrt{,25}$, $-\frac{5}{2} + \sqrt{12,25}$, and $-\frac{5}{2} - \sqrt{12,25}$; or 3, 2, 1, and -6.

Again, from the Equations $B^2 = 2A + \frac{1}{4}a^2 - b$, $2BC = aA - c$, and $C^2 = A^2 - d$; by supposing $B = 0$, we have $2A + \frac{1}{4}a^2 - b = 0$, and also $aA - c = 0$, hence $A = \frac{c}{a}$ and from the Equation $C^2 = A^2 - d$, we get $C = \sqrt{A^2 - d}$. write $\frac{c}{a}$ for A in the Equation $2A + \frac{1}{4}a^2 - b = 0$, and

you

you will have $\frac{2c}{a}+\frac{1}{4}a^2-b=0$, or $\frac{2c}{a}+\frac{1}{4}a^2=b$.

Hence you are to observe, that a Biquadratic may be solved by a Quadratic; when the Co-efficient of the third Term is equal to $\frac{1}{4}$ of the Square of that of the second Term, plus twice that of the fourth Term divided by that of the second.

EXAMPLE.

Let $x^4-12x^3+40x^2-24x-837=0$.

Here $a=-12$, $b=40$, $c=-24$, and $d=-837$, hence $A=\frac{c}{a}=\frac{-24}{-12}=2$, therefore $C=\sqrt{A^2-d}=\sqrt{4+837}=\sqrt{841}=29$, and by writing 12, 2, and 29 respectively for a, A and C, in the second general Theorem, you will have $x=3\pm\sqrt{9\pm29-2}$; so that two of the Roots are $3\pm\sqrt{36}$, or 9 and -3, the other two Roots are $3\pm\sqrt{-22}$ both impossible.

Lastly, If C be supposed $=0$, in the Equations $2BC=aA-c$, and $C^2=A^2-d$, then will $aA-c=0$, or $A=\frac{c}{a}$, and $A^2-d=0$, or $A=\sqrt{d}$, therefore $\frac{c}{a}\sqrt{d}$, and from the Equation $B^2=2A+\frac{1}{4}a^2-b$, we get $B=\sqrt{2A+\frac{1}{4}a^2-b}$: But $\frac{a}{c}=\sqrt{d}$, therefore $\frac{c^2}{a^2}=d$.

Hence you are to observe, that if the Square of the Co-efficient of the fourth Term divided by the Square of that of the second Term, be equal to the last Term, the Equation may be solved by a Quadratic.

EXAMPLE.

Given $x^4+12x^3-316x^2+288x+576=0$.

Here $a=12$, $b=-316$, $c=288$, and $d=\frac{c^2}{a^2}=576$; hence

hence $A=\frac{c}{a}=\frac{288}{12}=24$, and $B=\sqrt{2A+\frac{1}{4}a^2-b}$ $=\sqrt{400}=20$; and by writing 12, 24, and 20, respectively for a, A and B, in the first general Theorem, you will find $x=\pm 10-3\pm\sqrt{9\mp 60+100-24}=\pm 10-3\mp\sqrt{85\mp 60}$. Therefore the four Roots of the given Equation are 2, 12, $-13+\sqrt{145}$, and $-13-\sqrt{145}$.

A Biquadratic may be unfolded by Division, and solved by a Quadratic.

When the Difference between the Co-efficient of the third Term and the Square of half that of the second Term is equal to the Co-efficient of the fourth Term divided by half that of the second Term.

EXAMPLE I.

Given $x^4+10ax^3+27a^2x^2+10a^3x=b$, to find x.

Here, dividing by x^2+5ax, we have $x^2+5ax+2a^2=\frac{b}{x^2+5ax}$, or $\overline{x^2+5ax}|^2+2a^2\times\overline{x^2+5ax}=b$, and by completing the Square, &c. $x^2+5ax+a^2=\sqrt{a^4+b}$, hence $x=\sqrt{5{,}25a^2+\sqrt{a^4+b}}-2{,}25a$.

EXAMPLE II.

Given $x^4-4ax^3+5a^2x^2-2a^3x=b$, to find x.

Here, dividing by x^2-2ax, you will discover that $\overline{x^2-2ax}|^2+a^2\times\overline{x^2-2ax}=b$; hence $x^2-2ax+\frac{1}{2}a^2=\sqrt{\frac{1}{4}a^4+b}$, and $x=\sqrt{\frac{1}{2}a^2+\sqrt{\frac{1}{4}a^4+b}}+a$.

SCHOLIUM.

The Sign of the second Term of the Divisor must be the same as that of the second Term of the proposed Biquadratic, and its Co-efficient must always be half that of the second Term of the Equation to be unfolded.

If the third Term be wanting, the Division will succeed, if the Co-efficient of the fourth Term divided by the Cube of half that of the second Term gives -1, in the Quotient, as in the following Example, wherein let there be given $x^4-2ax^3+a^3x=b$, to find x.

Here dividing by x^2-ax, we have $x^2-ax-a^2=\frac{b}{x^2-ax}$, or $\overline{x^2-ax}|^2-a^2\times\overline{x^2-ax}=b$; and $x=\sqrt{\tfrac{1}{4}a^2+\sqrt{\tfrac{1}{4}a^4+b}}+\tfrac{1}{2}a$.

Sometimes Equations of six or eight, &c. Dimensions may be unfolded by Division, and reduced to Cubics, Biquadratics, &c. respectively.

Given $x^6+12x^5+60x^4+162x^3+252x^2+216x=118255$, to find x.

Here, dividing by x^3+6x^2+12x, we have $x^3+6x^2+12x+18=\frac{118255}{x^3+6x^2+12x}$, or $\overline{x^3+6x^2+12x}|^2+18\times\overline{x^3+6x^2+12x}=118255$; and by completing the Square, &c. we have $x^3+6x^2+12x+9=\sqrt{118336}=344$, or $x^3+6x^2+12x+8=343$, here by extracting the Cube Root on both Sides, we get $x+2=7$, and $x=5$.

In like Manner if there be given $x^{4n}-2ax^{3n}-3a^2x^{2n}+4a^3x^n=b$, to find x.

Then dividing by $x^{2n}-ax^n$, you will find that $\overline{x^{2n}-ax^n}|^2-4a^2\times\overline{x^{2n}-ax^n}=b$; hence $x^{2n}-ax^n-2a^2=\sqrt{4a^4+b}$, or $x^{2n}-ax^n=2a^2+\sqrt{4a^4+b}$, and $x^n-\tfrac{1}{2}a=\sqrt{2{,}25a^2+\sqrt{4a^4+b}}$, and consequently $x=\sqrt[n]{\tfrac{1}{2}a+\sqrt{2{,}25a^2+\sqrt{4a^4+b}}}$.

What is here delivered, may suffice at present for unfolding Equations by Division; I shall now demonstrate other Properties of another Class of Biquadratics.

When the Co-efficient of the fourth Term divided by that of the second Term gives the Square Root of the last Term in the Quotient, then the Square may always be completed

completed by adding a certain Quantity to both Sides of the proposed Biquadratic.

To find this Quantity, my Rule is this; Add the Square of half the Co-efficient of the second Term to twice the Square Root of the last Term, multiply that Sum by x^2, from the Product take the third Term, and the Remainder will be the Quantity to be added to both Sides of the given Equation.

EXAMPLE I.

Given $x^4 - 4ax^3 + 2a^2x^2 - 4a^3x + a^4 = 0$, to find x.

This Equation answers exactly to the foregoing Description; for if $-4a^3$ the Co-efficient of the fourth Term, be divided by $-4a$ the Co-efficient of the second Term, the Quotient will be a^2 the Square Root of a^4 the last Term; to twice this Root a^2, add the Square of Half $-4a$ the Co-efficient of the second Term, and the Sum will be $6a^2$, which, being multiplied by x^2, and the third Term $2a^2x^2$ taken from the Product $6a^2x^2$, the Remainder is $4a^2x^2$, which, being added to both Sides of the given Equation, it becomes $x^4 - 4ax^3 + 6a^2x^2 - 4a^3x + a^4 = 4a^2x^2$; here by extracting the square Root, we have $x^2 - 2ax + a^2 = 2ax$, or $x^2 - 4ax = -a^2$; and $x = 2a \pm a\sqrt{3} = 2a \times \overline{1 + \frac{1}{2}\sqrt{3}}$.

EXAMPLE II.

Given $x^4 + 8x^3 - 21x^2 - 48x + 36 = 0$, to find x.

Here the Square Root of the last Term 36, must be -6 negative, because the Quotient of -48, the Co-efficient of the fourth Term divided by $+8$, that of the second Term is -6 negative, and by the preceding Rule we have $\overline{4^2 - 6 \times 2} \times x^2 + 21x^2$, or $25x^2$ for the Quantity to be added; here, then, by adding $25x^2$ to both Sides of the given Equation, it becomes $x^4 + 8x^3 + 4x^2 - 48x + 36 = 25x^2$; here, by extracting the Square Root we have $x^2 + 4x - 6 = 5x$, or $x^2 - x = 6$, and $x = \sqrt{6,25} + ,5 = 3$.

EXAMPLE

EXAMPLE III.

Given $x^4-4x^3+5x^2-4x+1=0$, to find x.

Here, by adding $\overline{2^2+1\times2}\times x^2-5x^2$, or x^2 to both Sides, and extracting the Square Root we have $x^2-2x+1=x$, or $x^2-3x=-1$; hence $x=1,5\pm\frac{1}{2}\sqrt{5}|$.

EXAMPLE IV.

Given $x^4-9x^3+15x^2-27x+9=0$, to find x.

Here, by adding $\overline{\frac{9}{2}\Big|^2+6}\times x^2-15x^2$, or $11,25x^2$ to both Sides, and extracting the Root, we get $x^2-4,5x+3=\sqrt{11,25|}\times x$, or $x^2-\overline{4,5+\sqrt{11,25|}}\times x=-3$; hence by completing the Square, &c. we have $x=\sqrt{2,8125|}\pm\sqrt{4,875+4,5\sqrt{2,8125|}|}+2,25$.

EXAMPLE V.

Given $2x^4+24x^3-315x^2+216x+162=0$, or $x^4+12x^3-157,5x^2+108x+81=0$, to find x.

Here, by adding $\overline{6^2+18}\times x^2+157,5x^2$, or $211,5x^2$ to both Sides, and extracting the Root, we have $x^2+6x+9=\sqrt{211,5|}\times x=2\sqrt{52,875|}\times x$, or $x^2+\overline{6-2\sqrt{52,875|}}\times x=-9$; and $x=\sqrt{52,875|}\pm\sqrt{52,875-6\sqrt{52,875|}|}-3$.

EXAMPLE VI.

Given $x^4-2ax^3+\overline{2a^2-c^2}\times x^2-2a^3x+a^4=0$, to find x.

Here (according to the Rule) add $3a^2\times x^2-\overline{2a^2-c^2}\times x^2$, or $a^2x^2+c^2x^2$ to both Sides, and extract the Root, and you

you will have $x^2 - ax + a^2 = \pm\sqrt{a^2+c^2} \times x$, or $x^2 - \overline{a+\sqrt{a^2+c^2}} \times x = -a^2$; here, by completing the Square, &c. we get $x = \pm\frac{1}{2}\sqrt{a^2+c^2} \pm \sqrt{\frac{1}{4}c^2 - \frac{1}{2}a^2 \pm \frac{1}{2}a\sqrt{a^2+c^2}} + \frac{1}{2}a$.

When the Square of half the Co-efficient of the fourth Term, divided by the Co-efficient of the third Term, less the Square of half that of the second Term, gives the last Term (a negative Quantity) in the Quotient; then you may complete the Square of the Biquadratic, by adding the Difference, multiplied by x^2, between the Co-efficient of the third Term and the Square of half that of the second Term to the fourth and fifth Terms with their Signs changed, and then adding this Sum to both Sides of the proposed Equation.

EXAMPLE I.

Given $x^4 + 6ax^3 + 8a^2x^2 - 2a^3x - a^4 = 0$, to find x.

This Equation being one of the Class above described, I therefore take $8a^2$ the Co-efficient of the third Term, from the Square of $3a$ half that of the second Term, and multiplying the Remainder $9a^2 - 8a^2$, or a^2, by x^2, the Product is a^2x^2 the first Term of the Quantity to be added: Hence, by adding $a^2x^2 + 2a^3x + a^4$, to both Sides of the given Equation, it becomes $x^4 + 6ax^3 + 9a^2x^2 = a^2x^2 + 2a^3x + a^4$; here, by extracting the Square Root, we have $x^2 + 3ax = ax + a^2$, or $x^2 + 2ax = a^2$, and $x = a\sqrt{2} - a$.

EXAMPLE II.

Given $x^4 - 8x^3 - 12x^2 + 84x - 63 = 0$, to find x.

Here multiplying the Difference between (-12) the Co-efficient of the third Term and the Square of (-4) half that of the second Term by x^2, the Product is $28x^2$; and adding $28x^2 - 84x + 63$ to both Sides of the given Equation, we have $x^4 - 8x^3 + 16x^2 = 28x^2 - 84x + 63$; Hence, by extracting the Square Root we get $x^2 - 4x = \pm$

$\overline{\sqrt{7}}\times\overline{x-1,5}$, or $x^2-\overline{4+2\sqrt{7}}\times x=-3\sqrt{7}$; and $x=\sqrt{7}\pm\sqrt{11+\sqrt{7}}+2$.

EXAMPLE III.

Given $x^4-2ax^3+\overline{a^2-b^2}\times x^2+2ab^2x-a^2b^2=0$.

Here, by the Rule, we have $\overline{a^2-a^2+b^2}\times x^2$, or b^2x^2 for the first Term of the Quantity to be added, and by adding $b^2x^2-2ab^2x+a^2b^2$, to both Sides of the given Equation, we have $x^4-2ax^3+a^2x^2=b^2x^2-2ab^2x+a^2b^2$; or $x^2-ax=bx-ab$, hence $x^2-\overline{a+b}\times x=-ab$; and $x=a$.

I shall now show another Method of reducing Numeral Biquadratics to Quadratics, which is comprehensive, though not quite so concise as some of the foregoing Rules; and that is, by solving the Cubic Equation (Page 200) derived from the general Solution of Biquadratics: When this Method will succeed, the Root A of the Cubic Equation will be always rational, and may therefore be readily obtained by the preceding Rules. To this Root A, add the two first Terms of the Square Root of the proposed Biquadratic, which Terms may always be had by bare Inspection; then, from the Square of the Sum take the Terms of the given Biquadratic, and the Remainder being added to both its Sides, will complete its Square.

EXAMPLE I.

Given $x^4-8x^3-6x^2+28x-4=0$, to find x.

Here $a=-8$, $b=-6$, $c=28$, and $d=-4$; these Numbers being written respectively for a, b, c, and d, in the cubic Equation $A^3-\frac{1}{2}bA^2+\overline{\frac{1}{4}ac-d}\times A+\frac{1}{2}bd-\frac{1}{8}a^2d-\frac{1}{8}c^2=0$, it becomes $A^3+3A^2-52A-54=0$: Here it is obvious $A=-1$; this Root $(-1,)$ being added to the two first Terms of the Square Root of the given Biquadratic, the Sum is x^2-4x-1; now from $\overline{x^2-4x-1}|^2$, taking $x^4-8x^3-6x^2+28x-4$, there remains $20x^2-20x+5$; this Remainder being added to both Sides of the gi-

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ven Equation, and the Square Root extracted we have $x^2-4x-1=\pm\sqrt{5}\times\overline{2x-1}$, or $x^2-\overline{4\pm2\sqrt{5}}\times x=\pm\sqrt{5}+1$, and $x=2\pm\sqrt{5}\pm\sqrt{10\pm3\sqrt{5}}$.

EXAMPLE II.

Given $x^4+16x-31=0$, to find x.

Here $a=0$, $b=0$, $c=16$, and $d=-31$, hence by writing 16 and -31 for c and d in the cubic Equation, we have $A^3+31A-32=0$, here it is plain by Inspection, that $A=1$; and because the Biquadratic wants its second and third Terms, I therefore take the Square Root of its first Term only, and taking $x^4+16-31$ from $\overline{x^2+1}^2$, the Remainder is $2x^2-16x+32$, which being added to both Sides of the given Equation, and the Square Root extracted, gives $x^2+1=\pm\sqrt{2}\times\overline{x-4}$, or $x^2+\sqrt{2}\times x=4\sqrt{2}-1$; and $x=\sqrt{4\sqrt{2}-\frac{1}{2}}-\frac{1}{2}\sqrt{2}$.

EXAMPLE III.

Given $x^4-6x^3-58x^2-114x-11=0$, to find x.

By writing $-6, -58, -114$ and -11 respectively for a, b, c, and d, in the cubic Equation it becomes $A^3+29A^2+182A-1256=0$: Here by trying the Divisors 1, 2, 4, 8, &c. of the last Term -1256, I find $A=4$; then from $\overline{x^2-3x+4}^2$ take the given Biquadratic, add the Remainder $75x^2+90x+27$, to both its Sides, and extract the Root, and you will have $x^2-3x+4=\pm\sqrt{3}\times\overline{5x+3}$, or $x^2-\overline{3\pm5\sqrt{3}}\times x=\pm3\sqrt{3}-4$; and $x=1,5\pm2,5\sqrt{3}\pm\sqrt{17\pm10,5\sqrt{3}}$.

EXAMPLE IV.

Given $z^4+2az^3-37a^2z^2-38a^3z+a^4=0$, to find z.

In order to reduce this Equation to a numeral one, write ax for z, and you will have $a^4x^4+2a^4x^3-37a^4x^2-38a^4x+a^4=0$, or $x^4+2x^3-37x^2-38x+1=0$: Now by

by writing 2, -37, -38 and 1 respectively for a, b, c, and d in the cubic Equation, there will arise $A^3+18,5A^2-20A-199,5=0$: Here, by proceeding as in the last Example, I find $A=-3$.

But this Equation has two more Rational Roots (as you may now readily find), namely -19 and $3,5$, and we may always use either of the three Rational Roots of the Cubic Equation, therefore I take that Root which brings out the most commodious Quadratic:

Thus, subtract $x^4+2x^3-37x^2-38x+1$; from $\overline{x^2+x-19}|^2$, add the Remainder 360, to both Sides of the numeral Equation $x^4+2x^3-37x^2-38x+1=0$; and extract the Root, then will $x^2+x-19=6\sqrt{10}$, or $x^2+x=19+6\sqrt{10}$, hence $x=\sqrt{19,25+6\sqrt{10}}-\frac{1}{2}$, and consequently $z\ (=ax)=a\sqrt{19,25+6\sqrt{10}}-\frac{1}{2}a$.

But the numeral Equation, and the original literal Equatiou may be both unfolded by Division, and each reduced to a Quadratic, independent of this Method:

Thus, dividing the given Equation by z^2+az, we have $z^2+az-38a^2=\frac{-a^4}{z^2+az}$, or $\overline{z^2+az}|^2-38a^2\times\overline{z^2+az}=-a^4$; here, by completing the Square, &c. we get $z^2+az-19a^2=6a^2\sqrt{10}$, or $z^2+az=19a^2+6a^2\sqrt{10}$, and $z=a\sqrt{19,25+6\sqrt{10}}-\frac{1}{2}a$, the very same as before.

The Resolution of literal Equations, in which the given and unknown Quantity are alike affected.

IN the Reduction of Equations of this Kind, there are two Cases.

CASE I.

If the Equation proposed be of even Dimensions, dividing it by the equal Powers of its two Quantities in the middle

middle Term, it will be reduced to an Equation of half the Dimensions of the given one.

CASE II.

If the Equation be of odd Dimensions, dividing it by the Sum of its two Quantities, it will be reduced to an Equation of even Dimensions, and may then be further reduced, by Case I.

EXAMPLE I.

Given $x^4-4ax^3+5a^2x^2-4a^3x+a^4=0$, to find x.

Here, dividing by a^2x^2, we have $\frac{x^2}{aa}-\frac{4x}{a}+5-\frac{4a}{x}+\frac{a^2}{xx}$, or $\frac{x^2}{aa}+2+\frac{a^2}{xx}-4\times\overline{\frac{x}{a}+\frac{a}{x}}+3=0$, or $\overline{\frac{x}{a}+\frac{a}{x}}|^2-4\times\overline{\frac{x}{a}+\frac{a}{x}}=-3$: Here, by completing the Square, &c. we get $\frac{x}{a}+\frac{a}{x}-2=1$, or $x^2-3ax=-a^2$; and $x=\frac{1}{2}a\times\overline{3\pm\sqrt{5}}|$.

Otherwise, by adding a^2x^2 to both Sides of the given Equation, and extracting the Root, we have $x^2-2ax+a^2=ax$, or $x^2-3ax=-a^2$, and consequently $x=\frac{1}{2}a\times\overline{3\pm\sqrt{5}}|$, as before.

EXAMPLE II.

Given $x^5-5ax^4-5a^2x^3-11a^3x^2-8a^4x+4a^5=0$.

This Equation corresponds to Case the second, I therefore divide it by $x+a$, and the Quotient is $x^4-6ax^3+a^2x^2-12a^3x+4a^4=0$: Here by adding $12a^2x^2$ to both Sides we have $x^4-6ax^3+13a^2x^2-12a^3x+4a^4=12a^2x^2$, or $x^2-3ax+2a^2=2a\sqrt{3}|\times x$, hence $x^2-\overline{3a+2a\sqrt{3}}|\times x=-2a^2$;

$-2a^2$; this solved, by completing the Square, &c. gives
$x = a \times \overline{1,5 + \sqrt{3}} \pm \sqrt{3\sqrt{3} + 3,25}$.

EXAMPLE III.

Given $x^6 - 24ax^5 + 195a^2x^4 - 685a^3x^3 + 195a^4x^2 - 24a^5x + a^6 = 0$, to find x.

Dividing by a^3x^3, and ranging the Terms, the given Equation becomes $\frac{x^3}{a^3} + \frac{a^3}{x^3} - 24 \times \overline{\frac{x^2}{a^2} + \frac{a^2}{x^2}} + 195 \times \overline{\frac{x}{a} + \frac{a}{x}} - 685 = 0$; but $\frac{x^3}{a^3} + \frac{a^3}{x^3} = \overline{\frac{x}{a} + \frac{a}{x}}\Big|^3 - 3 \times \overline{\frac{x}{a} + \frac{a}{x}}$, and $\frac{x^2}{a^2} + \frac{a^2}{x^2} = \overline{\frac{x}{a} + \frac{a}{x}}\Big|^2 - 2$, consequently $\overline{\frac{x}{a} + \frac{a}{x}}\Big|^3 - 24 \times \overline{\frac{x}{a} + \frac{a}{x}}\Big|^2 + 48 + 192 \times \overline{\frac{x}{a} + \frac{a}{x}} - 685 = 0$, or $\overline{\frac{x}{a} + \frac{a}{x}}\Big|^3 - 24 \times \overline{\frac{x}{a} + \frac{a}{x}}\Big|^2 + 192 \times \overline{\frac{x}{a} + \frac{a}{x}} = 637$; here, by completing the Cube, and extracting the Root, we have $\frac{x}{a} + \frac{a}{x} - 8 = \sqrt[3]{125} = 5$, or $\frac{x}{a} + \frac{a}{x} = 13$, hence $x^2 - 13ax = -a^2$; and $x = a \times \overline{6,5 \pm \sqrt{41,25}}$.

Of Approximation by converging Series.

SOME of the preceding Methods for finding the Roots of Equations, are confined to particular Cases; but that which I am here going to explain is universal, extend-

ing to the Resolution of all Equations whatever; and when a Numerical Equation is proposed to be solved by this Method, you are to make Trial with such Numbers as you find are near its Root, and in a few Trials you may easily get two Numbers, one of which will give the Equation a positive and the other a negative Result, and consequently the true Root, which makes it equal to nothing, will be between these Numbers.

If the Root be incommensurate, you may approach it to what Degree of exactness you please by repeating the Operation.

Put r for the assumed Root, and add to it a new unknown Quantity z; then write $r \pm z$ and its Powers for x and its Powers in the given Equation, and you will find a near Value of z by taking its first Power only, and with this Value the Operation is to be repeated.

EXAMPLE I.

Given $x^3 + 9x^2 + 4x - 80 = 0$.

Here, it is plain, that x is less than 3, and greater than 2; for by writing 3 instead of x in the given Equation, it becomes $27 + 81 + 12 - 80 = 40$, positive; but writing 2 for x, it becomes $8 + 36 + 8 - 80 = -28$, negative, and consequently x is (equal to) some Number between 2 and 3; but the Error arising by writing 2 for x, is less than that produced by writing 3, therefore the Root x is nearer 2 than 3, and is consequently less than $2\frac{1}{2}$. This being premised, assume $r = 2{,}4$; and put $a = 9$, $b = 4$, $s = 80$, then the given Equation will become $x^3 + ax^2 + bx - s = 0$: This Equation, by writing $r + z$ for x, will be transformed to $(z+r)^3 + a \times (z+r)^2 + bz + br - s = z^3 + 3rz^2 + 3r^2z + r^3 + az^2 + 2arz + ar^2 + bz + br = 0$; hence $z^3 + (3r + a) \times z^2 + (3r^2 + 2ar + b) \times z = s - r^3 - ar^2 - br$: Here by omitting all the Powers of z above the first, we have $(3r^2 + 2ar + b) \times z = s - r^3 - ar^2 - br$; hence $z = \frac{s - r^3 - ar^2 - br}{3r^2 + 2ar + b} =$

$$\frac{80 - (2{,}4)^3 - 9 \times (2{,}4)^2 - 4 \times 2{,}4}{3 \times (2{,}4)^2 + 2 \times 9 \times 2{,}4 + 4} = \frac{4{,}736}{64{,}48} = 0{,}7, \text{ which}$$

added

added to 2,4 gives 2,47 for the Value of x, once corrected; now, write 2,47 for r, and put $m=3r+a=16,41$ $n=3r^2+2ar+b=66,7627$, and $t=s-r^3-ar^2-br=0,142677$; then will the Equation $z^3+\overline{3r+a}\times z^2+\overline{3r^2+2ar+b}\times z=s-r^3-ar^2-br$; become $z^3+mz^2+nz=t$: Here, since z is very little, it is evident that $nz=t$ nearly, therefore $z=\frac{t}{n}=\frac{0,142677}{66,7627}=,00213$, nearly: Let $p=,00213$, and write p^2 for z^2; rejecting z^3 as inconsiderable, so shall the Equation $z^3+mz^2+nz=t$, become $mp^2+nz=t$, or $nz=t-mp^2$, hence $z=\frac{t-mp^2}{n}=\frac{0,142602549}{66,7627}=,0021359$, and therefore $x=r+z=2,4721359$; which is true to the last Figure.

EXAMPLE II.

Given $x^3-22x-24=0$.

By proceeding as in the first Example, I find that x is something greater than 5, therefore I put $r=5$; and since $a=0$, $b=-22$, and $s=-24$; therefore the general Cubic Equation $z^3+\overline{3r+a}\times z^2+\overline{3r^2+2ar+b}\times z=s-r^3-ar^2-br$, will be reduced to $z^3+3rz^2+\overline{3r^2-b}\times z=s-r^3+br$: Here, by taking only the first Power of z, we have $z=\frac{s-r^3+br}{3r^2-b}=\frac{24-125+110}{75-22}=\frac{9}{53}=0,16$; which, added to 5, gives 5,16, for the Value of x once corrected: Now put $r=5,16$; then (in this Case) m being $=3r=15,48$, $n=3r^2-b=57,8768$, and $t=s-r^3+br=0,131904$; we have $\frac{t}{n}=,00227=p$, and $z=\frac{t-mp^2}{n}=\frac{0,1318242331}{57,8768}=,00227766$, and consequently $x=r+z=5,16227766$; which is true to the last Figure.

EXAMPLE III.

Let $x^3-48x^2+200=0$,

Here it is obvious that x is less than 48, and by Trial, I find it is greater than 47; therefore I put $r=47$; and since $a=-48$, $b=0$, and $s=200$; therefore in this Case the Equation $z^3+\overline{3r+a}\times z^2+\overline{3r^2+2ar+b}\times z=s-r^3-ar^2-br$, becomes $z^3+\overline{3r-a}\times z^2+\overline{3r^2-2ar}\times z=ar^2-r^3-s$: Hence $z=\frac{ar^2-r^3-s}{3r^2-2ar}=\frac{2009}{2115}=0{,}9$, nearly, which, added to 47, gives 47,9, for the Value of x, once corrected; and by writing 47,9 for r in the Equation $z=\frac{ar^2-r^3-s}{3r^2-2ar}$, we have $z=\frac{ar^2-r^3-s}{3r^2-2ar}=\frac{29{,}441}{2284{,}83}=$,0128, which, added to 47,9, gives 47,9128, for the Value of x, twice corrected: Now put $r=47{,}9128$; then will $m=3r-a=95{,}7384$, $n=3r^2-2ar=2287{,}28041152$, and $t=ar^2-r^3-s=0{,}17949414848$: Hence $\frac{t}{n}=$,00007847 $=p$, and $z=\frac{t-mp^2}{n}=\frac{0{,}1794938253348}{2287{,}28041152}$ $=$,000078474779?, this added to 47,9128, gives 47,912878474779?, for the Root x, which is true in all its Places, and so are those in the two preceding Examples; the second Example is the most simple Form of a Cubic that can be proposed to be solved by this Method; but a Solution to that Equation would have failed by the Rule for Cubics, demonstrated at Page 178.

It has been shewn that substituting an affirmative Number for x, greater than the true Root, gives the Equation a positive Result; and since the Squares of all real Quantities are affirmative, it follows that the Sum of the Squares of the Roots of any Equation must exceed the Square of its greatest Root; and the Square Root of that Sum will therefore

therefore be greater than the greatest Root of the Equation.

Thus in the general Equation $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} +$, &c. $=0$, the Sum of the Squares of the Roots will (by what has been demonstrated on Pages 182, 183) be $p^2 - 2q$, whose Square Root $\sqrt{p^2 - 2q}$ is greater than the greatest Root of the Equation.

And for the same Reason, the Biquadratic Root of ($p^4 - 4p^2q + 4pr + 2q^2 - 4s$) the Sum of the fourth Powers of the Roots, will likewise exceed the greatest Root of the Equation; therefore no Number exceeding either of these Roots is to be substituted for x in the given Equation.

These Rules were invented by Sir Isaac Newton, for finding the Limits of Equations, in his Universal Arithmetic, Page 204, second Edition.

Moreover in estimating the Root of an Equation, having all its Terms, you need not try any Number which exceeds the greatest negative Co-efficient increased by Unity; for that Co-efficient so increased will always exceed the greatest Root of the Equation. (See Mr. Maclaurin's Algebra, third Edition, Page 192)

But it is to be observed, that the greatest negative Co-efficient will often greatly exceed the greatest positive Root of the Equation, and sometimes it will be very near the Root, and since this Limit is had by Inspection, I thought it might not be amiss to mention it.

In solving Biquadratic and higher Equations, find the Root true, to three or four decimal Places; then substitute the Powers of the Decimals, last found, respectively, for those of z above the third Power, and solve the Cubic Equation thence arising as before.

EXAMPLE I.

Given $x^4 - 38x^3 + 210x^2 + 538x + 289 = 0$.

Here the Square Root of the Sum of the Squares of the Roots is $\sqrt{\overline{38}|^2 - 2 \times 210}$, or 32, which shews that x is less than 32, and, by Trial, it appears to be less than

31;

31; but I find it is nearer 31 than 30; therefore I write $z+30,5$ for x, in the given Equation, and it becomes $\overline{z+30,5}|^4 - 38\times\overline{z+30,5}|^3 + 210\times\overline{z+30,5}|^2 + 538\times\overline{z+30,5}+289=0$, hence $z^4+84z^3+2314,5z^2+20790z=744,1875$; and $z=\frac{744,1875}{20790}=,035$, which, added to 30,5, gives 30,535, for the Value of x, once corrected: And by writing $\overline{,035}|^4$ for z^4, in the transformed Equation, we have $\overline{,035}|^4 + 84z^3 + 2314,5z^2 + 20790z = 744,1875$, hence $z^3 + 27,5535714\,z^2 + 247,5z = 8,8593749821354$.

Put $a=27,5535714$, $b=247,5$, $s=8,8593749821354$, and $r=,035$; then from the general Cubic Equation $z^3+\overline{3r+a}\times z^2+\overline{3r^2+2ar+b}\times z=s-r^3-ar^2-br$: (by putting $m=3r+a$, $n=3r^2+2ar+b$, and $t=s-r^3-ar^2-br$) we have $z^3+mz^2+nz=t$, and $z=\frac{t}{n}=,0006538=p$:

Therefore $z=\frac{t-mp^2}{n}=\frac{0,163067159391 25}{249,432424998}=$,00065375285 27, nearly; this added to 30,535, gives 30,53565375285 27, for the Root x, which is true in all its Places.

EXAMPLE II.

Given $x^5+6x^4-10x^3-112x^2-207x-110=0$.

Here the Square Root of the Sum of the Squares of the Roots is about $7\frac{1}{2}$, and, by Trials, x appears equal to $4\frac{1}{2}$ nearly: therefore I write $z+4,4$ for x, in the given Equation, and it becomes $z^5+28z^4+289,2z^3+1304,8z^2+2145,064z-142,94016=0$, or $z^5+28z^4+289,2z^3+1304,8z^2+2145,064z=142,94016$:

Hence $z=\frac{142,94016}{2145,064}=,06$. Now in order to get

two

two Decimal Figures more, take the four last Terms of the Equation, and you will have $289,2z^3+1304,8z^2+2145,064z=142,94016$, and by Division, $z^3+4,5z^2+7,417z=0,49426$.

Put $a=4,5$; $b=7,417$, $s=0,49426$, and $r=,06$, then from the general Cubic Equation (by taking only the first Power of z) we have $z=\frac{s-r^3-ar^2-br}{3r^2+2ar+b}=\frac{,032824}{7,9678}$ $=,00413$ and $,0641$, added to $4,4$, gives $4,4641$, for the Value of x, once corrected. Now, in the transformed Equation for z^5 and z^4, write the like Powers of $,0641$, and there will arise $\overline{,0641}|^5+28\times\overline{,0641}|^4+289,2z^3+1304,8z^2+2145,064z=142,94016$; hence $289,2z^3+1304,8z^2+2145,064z=142,9396862128$, or $z^3+4,5117565692z^2+7,4172337482z=0,4942589426$.

Here, $a=4,5117565692$, $b=7,4172337482$, and $s=0,4942589426$; put $r=,0641$, $m=3r+a$, $n=3r^2+2ar+b$, and $t=s-r^3-ar^2-br$; then will the general Cubic Equation become $z^3+mz^2+nz=t$; hence $z=\frac{t}{n}=\frac{,0000012934}{8,007967737}=,0000016151=p$: Here p and m being small, the Term mp^2, may be omitted in this Operation, and therefore by adding $,0000016151$, to $4,4641$, you will have $4,4641016151$, for the Value of x, twice corrected, which is true in every Figure.

SCHOLIUM.

If you assume a Number greater than the true Root, then the Value of z, from the transformed Equation will come out Negative, and must be added to the assumed Root at the end of each Operation; or else assume a new less Number for the Root, and find positive Values of z, as above.

Sometimes

Sometimes it happens by assuming the nearest whole Number to the Root, and taking only the first Power of z, that the first Decimal Figure exceeds the Truth, which Figure, being written for r, in the Equation $z = \frac{s-r^3-ar^2-br}{3r^2+2ar+b}$, will (for the most part) give z a negative Result; but, by writing a less Number for r, you may in a few Trials, easily get the Value of z true, to two or three Places of Decimals. For Instance: By substituting $4+z$ for x in the last given Equation, there will arise $z^5+26z^4+246z^3+984z^2+1233z-810=0$, or $z^5+26z^4+246z^3+984z^2+1233z=810$; whence $z=\frac{810}{1233}=0.6$; and by taking the four last Terms, we have $246z^3+984z^2+1233z=810$, or $z^3+4z^2+5.012195z=3.2926829$.

Here $a=4$, $b=5.012195$, $r=0,6$, and $s=3.2926829$; hence $z=\frac{s-r^3-ar^2-br}{3r^2+2ar+b}=\frac{-1.370634}{10.892195}=-12$, which, being negative, shews that z is less than 0,6, and 0.5 being written for r, gives z also a negative Result; but by writing 0,45 for r, we have $z=\frac{s-r^3-ar^2-br}{3r^2+2ar+b}=\frac{.13607015}{9.219695}=0.14$, therefore $z=0.464$, &c. Now if you write $,4+y$ for z, in the transformed Equation, and proceed as above, you will find $y=,0641016151$, therefore $z=.4+y=.4641016151$, and consequently, $x=4+z=4.4641016151$, the very same as before.

But a nearer Value of x might have been obtained by writing $.464+y$ for z.

Sometimes it may be convenient when the Co-efficients are large, to transform the given Equation to another whose Root shall be 10, or 100, Times, &c. less than that of the Equation proposed, and having (by repeated Trials) found the nearest (less) whole Number to the Root of the given Equation; then the first Decimal Figure of the Root of the transformed Equation, if it be

10 Times

10 Times less than that of the given one, will be known; if the Root be 100 Times less, then its two first Decimal Places will be determined, &c.

EXAMPLE

Given $x^6+240x^5+14200x^4-444000x^3-55230000x^2-926000000x=4300000000$. Here it will be easily found that the Value of x is between 50 and 60; but nearer to 60 than to 50, and in a few Trials, x will appear to be greater than 56, but less than 57; this being discovered, write $10y$ for x, in the given Equation, and it will become $1000000y^6+24000000y^5+142000000y^4-444000000y^3-552300000y^2-9260000000y=4300000000$ or $y^6+24y^5+142y^4-444y^3-5523y^2-9260y=4300$.

Here now we have got the first two figures of the Root (y) equal to 5.6; put $r=5.6$, and write $r+z$ for y in the transformed Equation, and there will arise $\overline{z+r}|^6+24\times\overline{z+r}|^5+142\times\overline{z+r}|^4-444\times\overline{z+r}|^3-5523\times\overline{z+r}|^2-9260\times z+r=4300$.

The compound Terms of this Equation being actually involved by the first Table of Powers in Involution (Page 97) and then r being put into Numbers, you will (after Transposition) have $z^6+57.6z^5+1284.4z^4+13775.52z^3+70636.104z^2+137918.62656z=4664.335104$, and $z=\frac{4664.335104}{137918.62656}=.033$, nearly; and by taking the four last Terms, we have $13775.52z^3+70636.104z^2+137918.62656z=4664.335104$, or $z^3+5,12765z^2+10.01186z=.33859$: Here $a=5.12765$, $b=10.01186$, $r=.033$, and $s=.33859$, hence $z=\frac{s-r^3-ar^2-br}{3r^2+2ar+b}=\frac{.0025787}{10.35355}=.000249$. Now, by writing .033249 for z, in the first three Terms of the second transformed Equation, then transposing the Result of .033249, and dividing by 13775 52, we get $z^3+5.127654273z^2+$ 10.011863549

$10.0118635 49z = .328595823$: Here $a = 5.127654273$, $b = 10.011863549$, $r = .033249$. and $s = .3385958 23$;

hence $z = \dfrac{s - r^3 - ar^2 - br}{3r^2 + 2ar + b} = \dfrac{.000006013934}{10.3561588} =$ $.0000005807108$.

Therefore the whole (increased) Value of z, is $.0332495807108$, and the first $r = 5.6$, therefore $y = r + z$ $= 5.6332495807108$, and consequently $x\ (= 10y) =$ 56.332495807108; which Root does not differ $.000000000000004$, or $\dfrac{1}{250000000000000}$ part of an Unit from the Truth.

But a nearer Value of x might have been obtained by writing p^2 for z^2 (in the last Operation) as in the preceding Examples.

When you find that the Square Root of the Sum of the Squares of the Roots differs considerably from the Root required; if the Root required exeeeds 9, make Trials with Multiples of 10, and having found the Root between two Numbers whose Difference is 10 (as in the the last Example) observe the Errors produced by each of those two Numbers, by which Means you may in two or three Trials with the intermediate Numbers, find the nearest (less) whole Number to the Root sought.

Having shown how to find the Roots of Equation accurate enough for most Purposes, with little Trouble; and pointed out how greater accuracy may be occasionally obtained, I shall now give an Instance of extracting the Roots of pure Powers.

EXAMPLE

Let it be required to extract the Sursolid Root of 7900.

The nearest Integer to the required Root is 6, which being something less than the Truth, I put x for the Defect

fect, and $r=6$: Then by involving $x+r$, to the fifth Power, we have $x^5+5rx^4+10r^2x^3+10r^3x^2+5r^4x+r^5=7900$:

Hence $x=\frac{7900-r^5}{5r^4}=\frac{124}{6480}=.019$: By writing 6.019 (every where) for r, and .019 for x in the two first Terms of the foregoing Equation, there will arise $\overline{,019}|^5+30.095\times\overline{.019}|^4+362.2836x^3+2180,5898048x^2+6562.4707037x+\overline{6.019}|^5=7900$, whence, by Transposition and Division, we get $x^3+6,01900016x^2+18,114180989x=.0002698519$: Here $a=6.01900016$, $b=18.114180989$, $r=\frac{.0002698519}{18.114180989}=.0000014897$, and $s=,0002698519$: Hence $x=\frac{s-r^3-ar^2-br}{3r^2+2ar+b}=\frac{.00000000003610063}{18.114360331975}=.00000000000199292$; therefore the whole Value of x is .0190014897199292; and, consequently $r+x=6.0190014897199292$, the Root required.

Or thus (without repeating the whole Operation), by writing ,019 for x in the two first Terms of the original Equation $x^5+5rx^4+10r^2x^3+10r^3x^2+5r^4x+r^5=7900$, and putting r ($=6$) into Numbers, we have $\overline{.019}|^5+30\times\overline{.019}|^4+360x^3+2160x^2+6480x+7776=7900$, hence by Transposition, and dividing by 360, we get $x^3+6x^2+18x=.344444433$. Here $a=6$, $b=18$, $r=,019$, and $s=.344444433$; hence $x=\frac{s-r^3-ar^2-br}{3r^2+2ar+b}=\frac{.000271574}{18.229083}=.000014897$.

Here the Root is brought out true, to nine Places of Decimals, with very little Trouble; and since the Roots of all pure Powers may be found in the same Manner, I shall therefore take no further Notice of them here, but shall now apply converging Series to extracting the Roots of Equations containing several Surds, whose Solutions by

by other Methods would be very laborious: But before we proceed to solving such Equation, it will be proper to observe, that if $a+b$ represents any Binomial, and if b be but small in comparison of a, then will,

$$\left.\begin{aligned}
&1.\ \frac{1}{a+b} = \frac{1}{a} - \frac{b}{a^2} = \frac{1}{a} \times \overline{1 - \frac{b}{a}} \\
&2.\ \overline{a+b}|^{\frac{1}{2}} = a^{\frac{1}{2}} + \frac{b}{2a^{\frac{1}{2}}} = a^{\frac{1}{2}} + \frac{a^{\frac{1}{2}}b}{2a} \\
&3.\ \frac{1}{\overline{a+b}|^{\frac{1}{2}}} = \frac{1}{a^{\frac{1}{2}}} - \frac{b}{2a^{\frac{3}{2}}} = \frac{1}{a^{\frac{1}{2}}} - \frac{b}{2a \times a^{\frac{1}{2}}} \\
&4.\ \overline{a+b}|^{\frac{1}{3}} = a^{\frac{1}{3}} + \frac{b}{3a^{\frac{2}{3}}} = a^{\frac{1}{3}} + \frac{a^{\frac{1}{3}}b}{3a} \\
&5.\ \frac{1}{\overline{a+b}|^{\frac{1}{3}}} = \frac{1}{a^{\frac{1}{3}}} - \frac{b}{3a^{\frac{4}{3}}} = \frac{1}{a^{\frac{1}{3}}} - \frac{b}{3a \times a^{\frac{1}{3}}} \\
&6.\ \overline{a+b}|^{\frac{1}{4}} = a^{\frac{1}{4}} + \frac{b}{4a^{\frac{3}{4}}} = a^{\frac{1}{4}} + \frac{a^{\frac{1}{4}}b}{4a} \\
&7.\ \frac{1}{\overline{a+b}|^{\frac{1}{4}}} = \frac{1}{a^{\frac{1}{4}}} - \frac{b}{4a^{\frac{5}{4}}} = \frac{1}{a^{\frac{1}{4}}} - \frac{b}{4a \times a^{\frac{1}{4}}}
\end{aligned}\right\}\text{nearly.}$$

These Theorems and Extractions may be continued on at Pleasure by proceeding as on Pages 107, 108; but the two first Terms of the Roots as they are here exhibited, will, in most Cases, be sufficiently near the Truth.

EXAMPLE II.

Given $\sqrt{1+x^2} + \sqrt{2+x^2} + \sqrt{3+x^2} = 10$, to find the Value of x.

Here it appears by Inspection that x is about 3; I therefore write $3+e$ for x in the given Equation, rejecting all the

the Powers of e above the first, on account of their smallness; and there arises $\sqrt{10+6e}+\sqrt{11+6e}+\sqrt{12+6e}=10$: Now, because $6e$ is little in respect of 10, 11, and of 12, therefore by Theorem II. we have $\sqrt{a+b}=\sqrt{10+6e}=\sqrt{10}+\frac{6e}{2\sqrt{10}}=\sqrt{10}+\frac{3\sqrt{10}\times e}{10}=3.16227+.94868e$, nearly: Likewise $\sqrt{11+6e}=\sqrt{11}+\frac{3\sqrt{11}\times e}{11}=3.31662+.90453e$, and $\sqrt{12+6e}=\sqrt{12}+\frac{3\sqrt{12}\times e}{12}=3.4641+.86602e$: Hence the Equation $\sqrt{10+6e}+\sqrt{11+6e}+\sqrt{12+6e}=10$, is reduced to $3.16227+.94868e+3.31662+.90453e+3.4641+.86602e=10$, or $2.71923e+9.94299=10$, hence $2.71923e=.05701$, and $e=.0209$; therefore $x=3.0209$, nearly, and, to repeat the Operation, let $3.0209+e$ be now written for x in the given Equation, and there will arise $\sqrt{10.12583681+6.0418e}+\sqrt{11.12583681+6,0418e}+\sqrt{12.12583681+6.0418e}=10$: Whence, by Theorem II. we have $\sqrt{10.12583681}+\frac{6.0418e}{2\sqrt{10.12583681}}+\sqrt{11.12583681}+\frac{6.0418e}{2\sqrt{11.12583681}}+\sqrt{12.12583681}+\frac{6.0418e}{2\sqrt{12.12583681}}=10$; or $3.182112+.94933e+3.33554145+.90567e+3.48221722+.86752e=10$, hence $2.72252e+9.99987067=10$, and $2.72252e=.00012933$; therefore $e=.0000475$, and consequently $x=3.0209475$.

EXAMPLE II.

Let $\frac{20x}{\sqrt{16+5x+x^2}} + \frac{x\sqrt{5+x^2}}{25} = 34.$

Here by a few Trials I find that x is something greater than 20, therefore I write $20+e$ for x in the proposed Equation (rejecting the second Powers of e, as before) and there arises $\frac{400+20e}{\sqrt{516+45e}} + \frac{20+e}{25} \times \sqrt{405+40e}$ $=34.$

But, by Theorem the 3. $\frac{1}{\sqrt{516+45e}} \left(= \frac{1}{\sqrt{516}} - \frac{45e}{1032\times\sqrt{516}}\right) = .044022545 - .0019195e$, nearly; this, multiplied by $400+20e$ (reserving only the first Power of e) gives $\frac{400+20e}{\sqrt{516+45e}} = 17.609018 + .11265e.$

And, by Theorem the 2, we have $\sqrt{405+40e}\left(= \sqrt{405} + \frac{20e}{\sqrt{405}}\right) = 20.12461179 + .9938079e$, nearly; this Equation, multiplied by $\frac{20+e}{25} = .8 + ,04e$, omitting the second Power of e) gives $\frac{20+e}{25} \times \sqrt{405+40e}$ $=16.099689 + 1.60003e$: Now by writing these Values for their Equals, in the Equation $\frac{400+20e}{\sqrt{516+45e}} + \frac{20+e}{25}$ $\times \sqrt{405+40e} = 34$, you will have $17,609018 + .11265e$ $+ 16.099689 + 1.60003e + 34$, or $1.71268e = .291293$, hence $e = .1700$, and therefore $x = 20.1700.$

EXAMPLE III.

Given $\overline{1+x}|^{\frac{1}{2}} + \overline{1+x^2}|^{\frac{1}{3}} + \overline{1+x^3}|^{\frac{1}{4}} = 6.5.$

Here

Here x appears to be greater than 3, I therefore write $3+e$ for x in the given Equation (retaining only the first Power of e) and there arises $\overline{4+e}|^{\frac{1}{2}} + \overline{10+6e}|^{\frac{1}{3}} + \overline{28+27e}|^{\frac{1}{4}} = 6.5$.

This Equation, by the preceding Theorems, becomes

$$2+\frac{e}{4}+\overline{10}|^{\frac{1}{3}}+\frac{\overline{10}|^{\frac{1}{3}}\times 2e}{10}+\overline{28}|^{\frac{1}{4}}+\frac{\overline{28}|^{\frac{1}{4}}\times 27e}{4\times 28}=6.5,$$

that is, $2+.25e+2.15443469+.4308869e+2.30032663+.554543e=6.5$; hence $1.23543e=.04523868$, and $e=.036$, and consequently $x=3.036$.

When greater Exactness is necessary, it may be easily obtained by repeating the Operation, as in the first of these three Examples; so that what is here delivered, will be sufficient to show how to solve such Equations when they happen to occur in the Resolution of Problems; but it is to be observed, as all the Powers of e above the first are rejected in the Process, that this Method only doubles the Number of Figures at each Operation.

If there be two, or more Equations, and as many unknown Quantities, exterminate those Quantities till you get an Equation containing but one unknown Quantity, whose Value being found by some of the preceding Methods, the rest of those Quantities may be readily determined, as will appear by what follows.

EXAMPLE I.

Given $\left\{\begin{array}{r} e-x=10 \\ xy\ +ex=900 \\ exy=3000 \end{array}\right\}$ to find e, x, and y.

By the first Equation $e=10+x$, this Value being substituted for e in the second Equation, we have $xy+10x+x^2=900$, hence $y=\frac{900-10x-x^2}{x}$, this Value being written for y and $10+x$, for e in the third Equation there arises $9000+800x-20x^2-x^3=3000$, or $x^3+20x^2-800x=6000$. Here, by trying with Multiples of 10; I soon find

find that x is between 20 and 30, nnd in a very few Trials more, I find that x is greater than 23.9 and less than 24; put $a=20$, $b=800$, $r=23.9$, and $s=6000$; and write $z+r$ for x in the Equation $x^3+ax^2-bx=s$ $(=x^3+20x^2-800x=6000)$, and it will be transformed to

$$\left.\begin{array}{r} z^3+3rz^2+3r^2z+r^3 \\ +\ az^2+2arz+ar^2 \\ -bz-br \end{array}\right\}=s, \text{ or } z^3+\overline{3r+a}$$

$$\times z^2+\overline{3r^2+2ar-b}\times z=s-r^3-ar^2+br; \text{ hence } z=$$
$$\frac{s-r^3-ar^2+br}{3r^2+2ar-b}=\frac{43.881}{1869.63}=.0234, \text{ nearly.}$$

Now, by putting $r=23.9234$, and proceeding as in the foregoing Solutions of Cubic Equations, the whole increased Value of z will be $.02344345629 65$; therefore $x=23.9234434562965$; consequently $e\ (=10+x)=$ 33.9234434562965, and $y=\dfrac{900-10x-x^2}{x}=$ $\dfrac{88.4344186307 31}{23.9234434562965}=3.6965589333595.$

EXAMPLE II.

Given $\left\{\begin{array}{l} x+xy=40 \\ x^2y-y^2=63 \end{array}\right\}$, to find x and y.

The first Equation divided by $1+y$, gives $x=\dfrac{40}{1+y}$; this Value being substituted for x, in the second Equation there arises $\dfrac{1600y}{1+2y+y^2}-y^2=63$, this Equation, multiplied by $1+2y+y^2$, becomes $1600y-y^2-2y^3-y^4=63+126y+63y^2$, hence $y^4+2y^3+64y^2-1474y=-63$: Here it is evident at first sight, that y is less than 10; and, by Trial, I find $y=9$, whence $x=\dfrac{40}{1+y}=4$.

EXAMPLE

EXAMPLE III.

Given $\begin{cases} x+yz=10=a \\ y+xz=11=b \\ z+xy=14=c \end{cases}$

From the first Equation we get $x=a-yz$, and from the second, $x=\frac{b-y}{z}$; therefore $\frac{b-y}{z}=a-yz$, hence $y=\frac{az-b}{z^2-1}$, z Times this Equation taken from the fourth ($x=a-yz$), leaves $x=a-\frac{az^2-bz}{z^2-1}=\frac{bz-a}{z^2-1}$: Now by writing $\frac{bz-a}{z^2-1}$ for x and $\frac{z^2-b}{z^2-1}$ for y, in the third given Equation, there will arise $z+\overline{\frac{bz-a}{z^2-1}}\times\overline{\frac{az-b}{z^2-1}}=c$, hence $z^5-2z^3+z+abz^2-a^2z-b^2z-ab=c-2cz^2+cz^4$, or $z^5-cz^4-2z^3+\overline{ab+2c}\times z^2+\overline{1-a^2-b^2}\times z+ab-c=0$.

This Equation in Numbers, becomes $z^5-14z^4-2z^3+138z^2-220z+96=0$; here if the Root ($z$) be incommensurable, it may be easily obtained by converging Series; but in order to estimate the Root nearly, it may be observed, that we have found $x=\frac{bz-a}{z^2-1}$, and $y=\frac{az-b}{z^2-1}$, which shew that x is greater than y, because b is greater than a: In like Manner, from the second and third given Equations, you will find $y=\frac{cx-b}{x^2-1}$, and $z=\frac{bx-c}{x^2-1}$; which shew that y is greater than z, therefore it is evident by the first given Equation, that z must be less than 3, and by Trial in the Equation $z^5-14z^4-2z^3+138z^2-220z+96=0$, I find $z=2$, therefore $y=\frac{bz-a}{z^2-1}=3$, and $x=a-yz=4$.

In this Example, you may observe, that the Square Root of the Sum of the Squares of the Roots is very great in Respect of the Root (2), which will evidently be the Case, when the greatest positive Root of an Equation is very small in respect of the negative ones whose Squares are likewise affirmative: But by observing the Proportion of x, y, and z to each other, and the given Equations, we easily found a very near limit of z, which it could not surpass; and there may be frequently some Observations made in the Course of the Resolution of Equations by which a near Value of some of the unknown Quantities may be discovered, so that when the Roots are Integers, they may be readily found.

EXAMPLE IV.

$$\text{Given}\left\{\begin{aligned} x+yz&=11\\ xy+x^2z&=85\\ xyz+yz^2&=48\end{aligned}\right\}$$

From the first Equation we get $y=\frac{11-x}{z}$, and from the third, $y=\frac{48}{xz+z^2}$, therefore $\frac{11-x}{z}=\frac{48}{xz+z^2}$ hence $11x-x^2+11z-xz=48$, and $z=\frac{48+x^2-11x}{11-x}$; this Value, being wrote for z, in the Equation $y=\frac{11-x}{z}$ gives $y=\frac{121-22x+x^2}{48+x^2-11x}$; and by substituting the Values of y and z, in the second Equation, there will arise $\frac{121x-22x^2+x^3}{48+x^2-11x}+\frac{48x^2+x^4-11x^3}{11-x}=85.$

This Equation multiplied by $\overline{48+x^2-11x}\times\overline{11-x}$, gives $x^6-22x^5+216x^4-1023x^3+1941x^2+1331x=44880-85x^3+1870x^2-14365x$, or $x^6-22x^5+216x^4-938x^3+71x^2+15696x=44880$. Here the Square Root of the Sum of the Squares of the Roots is about 7, and, by Trial,

Trial, I find $x=5$; consequently $z=\frac{48+x^2-11x}{11-x}=$ 3, and $y=\frac{11-x}{z}=\frac{6}{3}=2$.

Sometimes the unknown Quantities are so involved as to render this Way of solving Equations impracticable, or very tedious, in which Cases the following Method may be of Use.

Let the Values of the unknown Quantities be taken pretty near the Truth, (which, from the Nature of the Question may always be done) then write the assumed Values connected to new unknown Quantities respectively for those in the proposed Equations, rejecting all their Powers above the first.

EXAMPLE I.

Given $\left\{\begin{matrix} x^4+y^4=10000 \\ x^5-y^5=25000 \end{matrix}\right\}$, to find x and y.

From the second Equation it is obvious that y is less than x, and from the first, it appears that x is less than 10; I therefore put $r=9$, and $s=8$; then by writing $e+r$ for x, and $z+s$ for y, in the given Equations, rejecting all the Terms that would arise above the first Powers of e and z, we have $4r^3e+4s^3z+r^4+s^4=10000$, and $5r^4e-5s^4z+r^5-s^5=25000$; or $4r^3e+4s^3z=10000-r^4-s^4=-657$, and $5r^4e-5s^4z=25000-r^5+s^5=-1282$.

Put $a=4r^3=2916$, $b=4s^3=2048$, $c=5r^4=32805$, $d=5s^4=-20480$, $m=-657$, and $n=-1282$, then the transformed Equations will become $ae+bz=m$; and $ce+dz=n$, hence $e=\frac{n-dz}{c}$, this Value being wrote for e, in the Equation $ae+bz=m$, we have $\frac{an-adz}{c}+bz=m$, or $an-adz+bcz=cm$, hence $z=\frac{cm-an}{bc-ad}=\frac{-17814573}{126904320}=-.14$, and $e=\frac{n-dz}{c}=\frac{-4149,2}{32805}=$

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-13, nearly, therefore $x=8{,}87$, and $y=7{,}86$, nearly.

Now in order to repeat the Operation, put $r=8{,}87$, and $s=7{,}86$; then will $a=4r^3=2791$, $b=4s^3=1942$, $c=5r^4=30950$, $d=5s^4=-19083{,}59$, $m=10000-r^4-s^4=-6{,}77$, and $n=25000-r^5+s^5=94$. Whence $z=\frac{cm-an}{bc-ad}=\frac{-470769}{113365553}=-.00415$, and $e=\frac{n-dz}{c}=\frac{14{,}40555}{30950}=.00047$, nearly; therefore $x=8{,}87047$, and $y=7{,}85585$.

It is very easy to see that this Method may be extended to the Resolution of three or four Equations, including as many unknown Quantities; but as these Cases seldom happen in the Application of Algebra to the Resolution of Problems, and when they do, may, for the most Part be reduced to some of the Forms already treated of, I shall therefore subjoin only one Example more, wherein let there be given $\overline{20x+xy^2}|^{\frac{1}{3}}+\overline{8x}|^{\frac{1}{2}}=12$, and $\sqrt{x^2+y^2}+\frac{xy}{\sqrt{x^2-y^2}}=13$, to find x and y.

Here I assume $r=5$, and $s=4$; then by writing $r+e$ for x, and $s+z$ for y in the given Equations (rejecting the second Powers of e and z, and their Products) there will arise $\overline{20r+rs^2+20e+s^2e+2rsz}|^{\frac{1}{3}}+\overline{8r+8e}|^{\frac{1}{2}}=12$, and $\sqrt{r^2+s^2+2re+2sz}+\frac{rs+se+rz}{\sqrt{r^2-s^2+2rs-2sz}}=13$; or, in Numbers, $\overline{180+36e+40z}|^{\frac{1}{3}}+\overline{40+8e}|^{\frac{1}{2}}=12$, and $\sqrt{41+10e+8z}+\frac{20+4e+5z}{\sqrt{9+10e-8z}}=13$: But $\overline{180+36e+40z}|^{\frac{1}{3}}=\overline{180}|^{\frac{1}{3}}+\frac{\overline{180}|^{\frac{1}{3}}}{3\times180}\times\overline{36e+40z}=5.646216+.376414e+.418238z$, nearly, and $\overline{40+8e}|^{\frac{1}{2}}=\sqrt{40}+\frac{8e}{2\sqrt{40}}=6{,}324555+.632455e$, nearly, hence the

the first transformed Equation is reduced to $1.008869e + .418238z + 11.970771 = 12$, or to $1.008869e + .418238z = .029229$.

Again, $\sqrt{41 + 10e + 8z} = \sqrt{41} + \frac{10e + 8z}{2\sqrt{41}} = 6,403124 + .780868e + .624669z$, nearly, and $\frac{1}{\sqrt{9 + 10e - 8z}} = \frac{1}{3} - \frac{10e - 8z}{2 \times 9 \times 3} = .3333333 - .185185e + .148148z$, nearly, this Equation multiplied by $20 + 4e + 5z$, rejecting the Products of e into z and their second Powers, gives $\frac{20 + 4e + 5z}{\sqrt{9 + 10e - 8z}} = 6,666666 - 2.370367e + 4.629626z$, hence the second transformed Equation becomes $-1.5895e + 5.2543z + 13.06979 = 13$, or $1,5895e - 5.2543z = .06979$: Here $c = 1,5895$, $d = -5.2543$, $n = .06979$; and from the Equation $1.008869e + ,418238z = ,029229$, we have $a = 1,008869$, $b = ,418238$, and $m = .029229$. Hence $z = \frac{cm - an}{bc - ad} = \frac{-.02394947}{5.9656896} = -.0040$; and $e = \frac{n - dz}{c} = \frac{.0487728}{1.5895} = .0306$; therefore $x = 5.0306$, nearly, and $y = 3.996$.

If more exactness be desired, it may be obtained by writing 5.0306, for r, and 3.996 for s, in the transformed Equations, and then extracting the Roots, &c. as above.

Having (by converging Series) largely expatiated on finding the Values of unknown Quantities with determinate Powers, I shall now proceed to the Resolution of Exponential Equations.

EXAMPLE I.

Given $x^x = 100$, to find the Value of x.

By the Nature of Logarithms, we have $x \times \text{Log.}\ x =$ the

the Log. of 100=2: Here in a few Trials x will be found between 3 and 4; but nearer 4 than 3.

Assume $x=3.5$; then will the Log of x be =the Log. of 3,5=,544068, and $x\times$ Log. of $x=3.5\times.544068=$ 1.904238, which should be equal to 2, the Log. of 100, therefore the Error is (2—1,904238=) .095742, in Defect. Suppose $x=3.6$, then the Log. of $x=$ the Log. of 3.6 =,5563025; and $x\times$ Log. $x=3.6\times$,5563025= ,2.002689, therefore ,2.002689—2=.002689, Error in excess: Hence, as ,09842, the Sum of the Errors, is to ,1, the Difference of the assumed Numbers, so is ,002689 to ,00276; therefore $x=$ 3.6—,00276=,3.59724, nearly: Again, by taking $x=3,597$, we have Log. $x=$ the Log. of 3.597=.5559404, and $x\times$ Log. $x=3.597\times$,5559404 =1.9997176, which, subtracted from 2, leaves .0002824, the Error in Defect; and as ,0029714, the Sum of the Errors produced by the assumed Numbers 3.6, and 3.597, is to .003 the Difference of these Suppositions, so is ,0002824, to ,000285, this added to 3,597, gives 3.597285 for the Value of x twice corrected.

For the better understanding of this Operation (see the Rule which immediately precedes the Eighty-fourth Question in Arithmetic—Vide also the Solution to that Problem.

When the given Number in the proposed Equation is much too big for the Course of the Logarithmetic Tables, extract its Square Root, and the Square Root of that Root, and so on, till its integral Part is reduced to two or three Figures, the fewer the better.

For the Quantity required, substitute another, with a numeral Co-efficient corresponding to the Exponent of the Power of the Root you intend to subtract; the Value of this unknown Quantity being found, that of the original Equation will be known. The two next following Examples will render this quite easy.

EXAMPLE

EXAMPLE II.

Given $z^z=123456789$, to find z.

By putting $8x=z$, we have $\overline{8x}|^{8x}=123456789$, and by extracting the 8th Root $\overline{8x}|^x=10,2669009$; this Equation by Logarithms, will be reduced to $x\times$ Log. $8x=$ Log. $10.266909=1.0114392$: Here in a few Trials x will be found something greater than 1,08, for by writing 1,08 for x in the Quantity $x\times$ Log. $8x$, it becomes $1,08\times$ Log. $8,64=1,0114348$, which should have been equal to 1,0114392 (the Logarithm of 10,2669009) therefore the Error is .0000044 in Defect; and by writing 1.080003 for x, we have $1,080003\times$ Log. $8,640024=1,0114386$, which should have been equal to 1.0114392, hence the Error is ,0000006 in defect; whence as ,0000038, the Difference of the Errors, is to ,000003 the Difference of the Suppositions, so is ,0000006 the least Error to ,0000004, which added to 1,080003 gives 1,0800034 for the Value of x once corrected.

Again, by writing 1,0800034 for x, we have $1,0800034\times$ Log. $8,6400272=1,0114393$, therefore the Error is ,0000001 in excess; whence, as ,0000007 : 0,0000004 :: 0,0000001 : 0,00000005, this taken from 1,0800034, leaves 1,08000335 for the Value of x twice corrected: Hence $8x$ $(=z)$ $=8,6400268$.

EXAMPLE III.

Suppose $z^z=782757789696$, what is the Value of z?

Here I find that the Square Root of 782757789696 is 884736, the Cube of 96, I therefore put $6x=z$, whence $\overline{6x}|^{6x}=782757789696$, or $\overline{6x}|^{3x}=884736$; and $\overline{6x}|^x=96$: This Equation solved, by proceeding as in the last Example, gives $x=1,8826432$, and consequently $6x$ $(=z)$ $=11,2958592$.

SCHOLIUM.

When the given Number is neither a Square, Cube, &c. (as in the second Example) then by continuing the Extraction

Extraction of the Square Root to a sufficient Number of Decimal Places in each Operation, you may solve Equations containing very great Numbers: And if the given Quantity be a Square or Cube Number, as in the last Example, then the Extraction of the Root will be very easy, and the Value of the unknown Quantity will be readily obtained.

EXAMPLE IV.

Given $x^{3x}-21x^{2x}+147x^{x}=316$, to find x.

Put $a=7$, and $b=316$, then the given Equation will become $x^{3x}-3ax^{2x}+3a^2x^x=b$; whence by completing the Cube, and extracting the Root, we have $x^x-a=\sqrt[3]{b-a^3}$, or $x^x=\sqrt[3]{b-a^3}+a=4=2^2$, and consequently $x=2$.

EXAMPLE V.

Given $x^{x^x}=400$, to find x.

By Logarithms we have $x^x \times$ Log. $x=$ the Log of 400; and by assuming $x=2.33$, we have $x^x=\overline{2.33}|^{2.33}=7.176$; hence $x^x\times$ Log. $x=7.176\times.3673559=2.6361459$, from this Log. that of 400 being subtracted, there remains .0340859, the Error in excess: Again, suppose $x=2.324$, then will $x^x=7.0979$, and $x^x\times$ Log. $x=7.0979\times.3662361$ $=2.5995072$, this taken from 2.6020600 the Logarithm of 400, leaves .0025528 the Error in defect; and as .0366387, the Sum of the Errors is to .006 the Difference of the Suppositions, so is .0025528 to .000418, hence $x=2.324418$, nearly.

EXAMPLE VI.

Given $\left\{\begin{matrix} y^x=3000 \\ x^y=5000 \end{matrix}\right\}$, to find x and y.

By extracting the x Root of the first Equation we have $y=\overline{3000}|^{\frac{1}{x}}$, this Value substituted for y, in the second Equation,

Equation, gives $x^{\overline{3000}|^{\frac{1}{x}}} = 5000$. Here I assume $x = 4.7$, hence the Exponent $\overline{3000}|^{\frac{1}{x}}$ becomes $\overline{3000}|^{\frac{1}{4.7}}$: Now I divide 3,4771212, the Logarithm of 3000 by 4.7, and the Quotient .7398130, is the Log. of 5.493, therefore $\overline{3000}|^{\frac{1}{x}}$, or $\overline{3000}|^{\frac{1}{4.7}} = 5.493$: Hence the Equation $x^{\overline{3000}|^{\frac{1}{x}}} = 5000$, is reduced to $x^{5.493} = 5000$; and $5.493 \times \text{Log. } x = 5.493 \times .6720978 = 3.6918332$, this Log. taken from that of 5000, leaves .0071368, the Error in defect; therefore x is less than 4.7; for it is to be observed, that the greater x is assumed, the less the Result of $x^{\overline{3000}|^{\frac{1}{x}}}$ will be; therefore I assume $x = 4.69$; then $3000^{\frac{1}{x}} = \overline{3000}|^{\frac{1}{4.69}} = 5.513$; and $5.513 \times \text{Log. } x = 5.513 \times .6711728 = 3.7001756$, from this Log. that of 5000 being subtracted, there remains .0012056, the Error in excess, and as .0083424, the Sum of the Errors is to .01, the Difference of the Suppositions, so is .0012056, to .001445, therefore $x = 4.691445$, nearly; and $y = \overline{3000}|^{\frac{1}{x}} = \overline{3000}|^{\frac{1}{4.691445}} = 5.510132$.

You may find the Corrections by working with the Errors of the Numbers resulting from the Suppositions, instead of their corresponding Logarithms, as in the Solutions to the Eighty-fourth and Eighty-fifth Questions in the Arithmetical Part of this Book.

EXAMPLE

EXAMPLE VII.

Given $y^x = 9.5$, and $x^x + y^y = 32.25$, to find the Values of x and y.

By the first Equation we get $y = \overline{9.5}|^{\frac{1}{x}}$; this Value of y being substituted in the second Equation, there arises $x^x + \overline{9.5}|^{\frac{\overline{9.5}|^{\frac{1}{x}}}{x}} = 32.25$: Here in a few Trials I find that x is something less than 2.05, therefore I assume $x = 2.04$, then $x^x + 9.5^{\frac{\overline{9.5}|^{\frac{1}{x}}}{x}} = \overline{2.04}|^{2.04} + \overline{9.5}|^{1.4779} = 4.28198 + 27.85978 = 32.14176$, which should be equal to 32.25, therefore the Error is .10824, in defect: Suppose $x = 2.03$; then will $x^x + \overline{9.5}|^{\frac{\overline{9.5}|^{\frac{1}{x}}}{x}} = \overline{2.03}|^{2.03} + \overline{9.5}|^{1.49327} = 4.20936 + 28.84073 = 33.05009$, from this Number 32.25 being subtracted, there remains .80009, the Error in excess; whence as .90833 : 0.01 :: 0.10824 : 0.01191, this taken from 2.04, (the Supposition that produced the least Error) leaves 2.038809, for the Value of x; hence $y = \overline{9.5}|^{\frac{1}{x}} = \overline{9.5}|^{\frac{1}{2.038809}} = 3.016867$.

EXAMPLE VIII.

Given $x^x - x = y$, and $y^x + y = x$.

Add the two Equations together, and you will have $x^x + y^x - x + y = x + y$, or $y^x = 2x - x^x$, hence $y = \overline{2x - x^x}|^{\frac{1}{x}}$ therefore $x^x - x = \overline{2x - x^x}|^{\frac{1}{x}}$, and consequently $x^x - x -$

2x

$\overline{2x-x^x}|^{\frac{1}{x}}=0$: It appears by the fourth Equation, that $2x$ is greater than x^x, therefore it is evident that x is less than 2.

Assume $x=1.75$; then will the Term $\overline{2x-x^x}|^{\frac{1}{x}}=\overline{.83734}|^{\frac{1}{1.75}}$: Here because the Number under the Vinculum, is less than Unity, therefore (to augment it) I multiply it by $\overline{2}|^{1.75}$, that is, by 3.36358, and the Logarithm of the Product 2.8164600772, is .4497035, this divided by 1.75, the Quotient .2569734, is the Logarithm of 1.80706, which being divided by 2, gives $\overline{.83734}|^{\frac{1}{1.75}}=.90353$; hence $x^x-x-\overline{2x-x^x}|^{\frac{1}{x}}=2.66266-1.75-.90353=.00913$, which should be (o) nothing, therefore the Error is .00913, in excess.

Assume $x=1,748$, and multiply (.84163) the Number arising under the Vinculum, by $\overline{10}|^{1.748}$, that is, by 55.97576 (the Number corresponding to the Logarithm 1.748) then divide 1.6731212, the Logarithm of 47,1108 7888 the Product, by 1.748, and the Quotient .9571631, is the Logarithm of 9.06072, this divided by 10, gives $\overline{2x-x^x}|^{\frac{1}{x}}=\overline{.84163}|^{\frac{1}{1.748}}=.906072$, hence $x^x-x-\overline{2x-x^x}|^{\frac{1}{x}}=2.65437-1.748-.906072=.000298$, the Error in excess; and as .008832 : 0.002 :: 0,000298 : 0.000067, therefore $x=1.747933$; hence y $(=x^x-x=2.654097-1.747933)=.906164$.

EXAMPLE IX.

Given $x^{x+y}=y^8$, and $y^{x+y}=x^{18}$, to find x and y.

From the first Equation $y=x^{\frac{x+y}{8}}$, and from the second $y=x$

$y = x^{\frac{18}{x+y}}$, therefore $x^{\frac{x+y}{8}} = x^{\frac{18}{x+y}}$: Here because the Root x is involved in but one Term on each Side of the Equation, the Exponents $\frac{x+y}{8}$ and $\frac{18}{x+y}$, must necessarily be equal to each other; hence $\frac{x+y}{8} = \frac{18}{x+y}$, or $x^2 + 2xy + y^2 = 144$; therefore $x+y = \sqrt{144} = 12$, hence $y = 12 - x$: Now write 12 for $x+y$, and $12-x$ for y in the Equation $y = x^{\frac{x+y}{8}}$, and you will have $12 - x = x^{\frac{12}{8}}$, or $x^{\frac{3}{2}} = 12 - x$, hence $x^3 = 144 - 24x + x^2$, or $x^3 - x^2 + 24x = 144$; from this Cubic Equation you will easily find $x = 4$, and therefore $y = 12 - x = 8$.

EXAMPLE X.

Given $x^x - y^y = 4.35$, and $y^x - x = .25$.

By the second Equation $y = \overline{x + .25}|^{\frac{1}{x}}$; this Value of y substituted in the first Equation, gives $x^x - \overline{x+.25}|^{\frac{\overline{x+.5}|^{\frac{1}{x}}}{x}} = 4.35$: Assume $x = 2.25$, then will $x^x - \overline{x+.25}|^{\frac{\overline{x+.25}|^{\frac{1}{x}}}{x}} = \overline{2.25}|^{2.25} - \overline{2.5}|^{\frac{1.502665}{2.25}} = 6.20041 - 1.84401 = 4.35613$, which should be equal to 4.35, therefore the Error is .00613 in excess. Again, by assuming $x = 2.249$, and repeating the Operation, you will find the Error to be .00497 in defect; whence as 0111 : 0.001 :: 0.00497 : 0,0004477, therefore $x = 2.2494477$, nearly, and, consequently

quently $y=\overline{x+.25}|^{\frac{1}{x}} = \overline{2.4994477}|^{\frac{1}{2.2494477}} =$ 1.5026675.

EXAMPLE XI.

Given $x^y=y^x$, and $x^{y+n}=ax^{n+3}$, to find x and y, where a denotes a given Number.

The second Equation divided by x^n, gives $x^y=ax^3$, therefore $y^x=ax^3$; hence $y=\overline{ax^3}|^{\frac{1}{x}}$: This Value of y being substituted in the Equation $x^y=ax^3$, there arises $x^{\overline{ax^3}|^{\frac{1}{x}}}=ax^3$; or $x^{\overline{ax^3}|^{\frac{1}{x}}}-ax^3=0$: Here the Value of x may be readily found by Logarithms be a what it will.

If a be taken $=2$, then will $x=2$; for then will $x^{\overline{ax^3}|^{\frac{1}{x}}}-ax^3=2^{\overline{16}|^{\frac{1}{2}}}-16=\overline{2}|^4-16=16-16=0$; whence $y=\overline{ax^3}|^{\frac{1}{x}}=\overline{16}|^{\frac{1}{2}}=4$.

Thus I have introduced an easier Method of solving Exponential Equations than that of converging Series by Reversion, and by proceeding as in these Examples, the Roots of three or four Exponential Equations that can be reduced to an Equation comprizing but one unknown Quantity, may be approximated to any Degree of accuracy by repeating the Operations at the Equation so deduced. But as Equations may occur, from which an Equation containing but one unknown Quantity cannot be derived, without previously substituting for the unknown Quantities; I shall therefore subjoin another Example,

 and

and solve it by a Method which may be of Service in such Cases.

EXAMPLE XII.

Suppose $x^x + y^y = 285 = s$, and $y^x - x^y = 14 = d$; Quere, the Values of x and y.

Here, in a few Trials x will be found a little greater than 4, and y something less than 3; put $a = 4$, $c = 2.8$; $L = 1.3862943$, the Hyperbolical Logarithm of 4 $(=a)$, and $l = 1.0296193$, that of 2.8 $(=c)$; let $a + v = x$, and $c + z = y$, then since v and z are very small, we have $x^x = \overline{a+v}\,|^{a+v} = a^a \times \overline{1 + v + Lv}$, nearly, and $y^y = \overline{c+z}\,|^{c+z} = c^c \times \overline{1 + z + lz}$; likewise $y^x = \overline{c+z}\,|^{a+v} = c^a \times \overline{1 + lv} \times \overline{1 + \frac{az}{c}}$, and $x^y = \overline{a+v}\,|^{c+z} = a^c \times \overline{1 + Lz} \times \overline{1 + \frac{cv}{a}}$: Hence the given Equations are transformed to $a^a \times \overline{1 + v + Lv} + c^c \times \overline{1 + z + lz} = s$, and $c^a \times \overline{1 + lv} \times \overline{1 + \frac{az}{c}} - a^c \times \overline{1 + Lz} \times \overline{1 + \frac{cv}{a}} = d$. These, in Numbers, become $610.89134v + 36.26243z + 273.866618 = 285$, and $29.33409v + 20.56855z + 43.34116vz + 12.96265 = 14$: From the first of these Equations (by Transposition, and dividing by 610.89134,) we get $v = .018224 - .05936z$; this Value of v being substituted in the other Equation, gives $-2.57273z^2 + 19.61712z + 13.497234 = 14$; therefore $2.57273z^2 - 19.61712z = -.502766$, or $z^2 - 7.62502z = -.1954215$; hence $z = 3.81251 - \sqrt{14.339811} = .025716$: and $v = .018224 - .05936z = .016698$; therefore $x = 4.016698$, nearly, and $y = 2.825716$.

These Roots are very near the Truth, but greater Exactness may be obtained by writing 4.016698 and 2.825716 respectively for a and c; and their corresponding Logarithms for L and l in the transformed Equations.

The

The Hyperbolical Logarithm of any Number may be produced by multiplying the common Logarithm of the Number proposed by 2.3025851; thus the common Log. of 4 which is .6020600, being multiplied by 2.3025851, the Product 1.3862943, is the Hyperbolical Logarithm of 4; and thus you may furnish yourself with Hyperbolical Logarithms from those of the common Tables.

To find the Square Root of a Binomial when one of its Terms is Rational, the other a Surd Square Root.

EXAMPLE I.

Let it be proposed to extract the Square Root of $136-\sqrt{18432}$.

Put $A\pm\sqrt{B}=136\pm\sqrt{18432}$; and let the Root required be represented by $\sqrt{x}\pm\sqrt{y}$, then will $\sqrt{x}\pm\sqrt{y}=\overline{A\pm\sqrt{B}}|^{\frac{1}{2}}$; hence $x\pm2\sqrt{xy}+y=A\pm\sqrt{B}$; by equating the rational and irrational Parts of this Equation, we have $x+y=A$, and $2\sqrt{xy}=\sqrt{B}$, or $4xy=B$, this Equation taken from the Square of $x+y=A$, leaves $x^2-2xy+y^2=A^2-B$; hence $x-y=\sqrt{A^2-B}$, this Equation being added to, and subtracted from $x+y=A$, (and the Sum and Difference divided by 2) gives $x=\frac{A+\sqrt{A^2-B}}{2}=\frac{136+8}{2}=72$, and $y=\frac{A-\sqrt{A^2-B}}{2}=\frac{136-8}{2}=64$; and consequently $\sqrt{x}-\sqrt{y}=\sqrt{72}-\sqrt{64}=6\sqrt{2}-8$, the required Square Root of $136-\sqrt{18432}$.

EXAMPLE II.

What is the Square Root of $37-\sqrt{1200}$.

Here $A=37$, and $B=1200$; hence $x=\frac{A+\sqrt{A^2-B}}{2} = \frac{37+\sqrt{169}}{2} = 25$, and $y = \frac{A-\sqrt{A^2-B}}{2} = \frac{37-13}{2} = 12$, therefore $\sqrt{x}-\sqrt{y} = \sqrt{25}-\sqrt{12} = 5-2\sqrt{3}$, the Root sought.

It is obvious that this Method will always succeed, when A^2-B is a Square Number; otherwise it will be of no Service.

If the Sum and Difference of two Quantities x and y be raised to any Power n, and if the first, third, fifth, seventh, &c. Terms of that Power, collected into one Sum, be denoted by A, and the Rest of the Terms (in the even Places), by B; then the Difference of the Squares of A and B will be equal to the Difference of the Squares of x and y raised to the same Power n.

For the Co-efficients of the odd Terms in the n^{th} Power of $x+y$, put 1, a, c, e, &c. and for those of the even Terms put n, b, d, f, &c. then we shall have $x^n + ax^{n-2}y^2 + cx^{n-4}y^4$, &c. $= A$, and $nx^{n-1}y + bx^{n-3}y^3 + dx^{n-5}y^5$, &c. $= B$. This Equation added to, and subtracted from the first, gives $\overline{x+y}|^n = x^n + nx^{n-1}y + ax^{n-2}y^2 + bx^{n-3}y^3 + cx^{n-4}y^4 + dx^{n-5}y^5$, &c. $= A+B$; and $\overline{x-y}|^n = x^n - nx^{n-1}y + ax^{n-2}y^2 - bx^{n-3}y^3 + cx^{n-4}y^4 - dx^{n-5}y^5$, &c. $= A-B$, the n^{th} Power of $x+y$, and of $x-y$ respectively; and by multiplying these two Equations together, we have $\overline{x+y}|^n \times \overline{x-y}|^n = \overline{A+B} \times \overline{A-B} = A^2 - B^2$ $= \overline{\overline{x+y} \times \overline{x-y}}|^n = \overline{x^2-y^2}|^n$. Q. E. D.

EXAMPLE

EXAMPLE III.

Required the n^{th} Root of $A \pm \sqrt{B}$.

Let the required Root be represented by $x \pm \sqrt{y}$, then by proceeding as above, we have $x^n + ax^{n-2}y + cx^{n-4}y^2$, &c. $= A$, and $nx^{n-1}\sqrt{y} + bx^{n-3}y\sqrt{y} + dx^{n-5}y^2\sqrt{y}$, &c. $= \sqrt{B}$. Here, by extracting the n^{th} Root of the Sum, and also of the Difference of the first and second Equations, we have $x + \sqrt{y} = \left(A + \sqrt{B}\right)^{\frac{1}{n}}$, and $x - \sqrt{y} = \left(A - \sqrt{B}\right)^{\frac{1}{n}}$ half the Sum of these two Equations, gives $x = \frac{\left(A+\sqrt{B}\right)^{\frac{1}{n}}}{2} \pm \frac{\left(A-\sqrt{B}\right)^{\frac{1}{n}}}{2} = \frac{\left(A+\sqrt{B}\right)^{\frac{1}{n}}}{2}$ $\pm \frac{\left(A^2-B\right)^{\frac{1}{n}}}{2\times\left(A+\sqrt{B}\right)^{\frac{1}{n}}}$: and their Product gives $x^2 - y = \left(A^2-B\right)^{\frac{1}{n}}$, hence $y = x^2 \pm \left(A^2-B\right)^{\frac{1}{n}}$; whence y is likewise known.

To show the Use of these Conclusions, let it be proposed to extract the Cube Root of $45 + 29\sqrt{2} = 45 + \sqrt{1682}$.

Here $A = 45$, $B = 1682$, and $n = 3$; hence $x = \frac{\left(45+\sqrt{1682}\right)^{\frac{1}{3}}}{2} \pm \frac{343^{\frac{1}{3}}}{2\times\left(45+\sqrt{1682}\right)^{\frac{1}{3}}} = \frac{86.01219^{\frac{1}{3}}}{2}$ $\pm \frac{7}{2\times 86.01219^{\frac{1}{3}}} = \frac{4.414+1.586}{2} = 3$; and y $\left(= x^2 - \left(A^2-B\right)^{\frac{1}{n}}\right)$ $3^2 - 343^{\frac{1}{3}} = 9 - 7 = 2$, and therefore $x + \sqrt{y} = 3 + \sqrt{2}$, the Root required; for $3 + \sqrt{2}$

$\times\overline{3+\sqrt{2}}\times\overline{3+\sqrt{2}}=45+29\sqrt{2}$, the Binomial at first proposed.

EXAMPLE IV.

What is the Biquadratic Root of $161+\sqrt{25920}$?
Here $A=161$, $B=25920$, and $n=4$, hence $x=$

$$\frac{\overline{161+\sqrt{25920}}\,|^{\frac{1}{4}}}{2}\pm\frac{\overline{25921-25920}\,|^{\frac{1}{4}}}{2\times\overline{161+\sqrt{25920}}\,|^{\frac{1}{4}}}=$$

$$\frac{\overline{321.996896}\,|^{\frac{1}{4}}}{2}\pm\frac{1}{2\times\overline{321.99,\&c.}\,|^{\frac{1}{4}}}=\frac{4.236\pm.236}{2}$$

$=2$, and $y=4+\overline{25921-25920}\,|^{\frac{1}{4}}=5$, therefore $x+\sqrt{y}=2+\sqrt{5}$, the Root sought.

EXAMPLE V.

What is the surfolid Root of $76+\sqrt{5808}$.

Here, by proceeding as above, you will find $x=\frac{2.732\pm.732}{2}=1$, and $y=1-\overline{5776-5808}\,|^{\frac{1}{5}}=1-\sqrt[5]{-32}=3$, therefore the Root required is $1+\sqrt{3}$.

It is to be observed, that if neither the Sum nor Difference of the two Quantities between which the double Sign ($\pm$) is placed, will give x a whole or rational Number, then neither Term of the Root will be rational; in which Case this Method is of no Use: If the lower Sign ($-$) in the Value of x be taken, the lower Sign in ($x^2\pm\overline{A^2-B}\,|^{\frac{1}{n}}$) that of y must be taken accordingly; and if

x be

x be an Integer, its Value may be easily discovered, by extracting the n^{th} Root of $A+\sqrt{B}$, to two or three Decimal Places by Logarithms.

It may be likewise proper to remark here, that if A^2-B is not a perfect n^{th} Power, then this Method will be of no Service; unless we multiply the given Binomial by a Number that shall make it succeed; as in the following Example, wherein let it be required to extract the Cube Root of $22+\sqrt{486}$.

Here we have $\overline{A^2-B}|^{\frac{1}{n}} = \overline{484-486}|^{\frac{1}{3}} = \overline{-2}|^{\frac{1}{3}}$ irrational; but multiplying $22+\sqrt{486}$ by 2, the Product is $44+\sqrt{1944}$, hence $\overline{A^2-B}|^{\frac{1}{3}} = \overline{44^2-1944}|^{\frac{1}{3}} = \overline{-8}|^{\frac{1}{3}}$ rational; and, the Cube Root of $44+\sqrt{1944}$ divided by $\sqrt[3]{2}$, gives $\frac{2+\sqrt{6}}{\sqrt[3]{2}}$, the Root required.

SCHOLIUM.

As this Method brings out the Root with the (Surd) n^{th} Root of the Multiplier for its Denominator, therefore it is of but little Use, except when the n^{th} Root of the Multiplier is rational. Those who would find Multiplyers in a general Manner, may peruse Maclaurin's Algebra, p. 124, third Edition.

Sometimes in the Resolution of Cubic Equations there arise Binomials of this Form $A\pm\sqrt{-B^2q} = A\pm B\sqrt{-q}$ which, if the Root is expressible in rational Numbers, may be found as follows:

It being universally, $\overline{A^2-B^2q}|^{\frac{1}{n}} = x^2-y$, and, in the present Case $\overline{A^2+B^2q}|^{\frac{1}{3}}$ $(=x^2-y) = p^2+l^2\times q$; for putting $x\pm\sqrt{y} = p\pm l\sqrt{-q}$, we have $x^2=p^2$, and $y= -l^2q$, this Equation taken from $x^2=p^2$, leaves x^2-y (=

$\overline{A^2+B^2q}|^{\frac{1}{3}}$) $=p^2+l^2q$, as above, from ($\overline{A^2+B^2q}|^{\frac{1}{3}}=$) p^2+l^2q, subtracting p^2, the Square of some Divisor of A, the Remainder is l^2q, a known Multiple of the Square of l, a Divisor of B.

But $\overline{p+l\sqrt{-q}}|^3 = p\times\overline{p^2-3l^2q}+l\times\overline{3p^2-l^2q}\times\sqrt{-q}$; hence $A=p\times\overline{p^2-3l^2q}$, and $B=l\times\overline{3p^2-l^2q}$; Here it is evident that p and l are Divisors of A and B respectively, and the Signs of p and l must be taken, such as will give the Products of $p\times\overline{p^2-3l^2q}$, $l\times\overline{3p^2-l^2q}$ of the same Signs as A and B respectively.

EXAMPLE I.

What is the Cube Root of $81+\sqrt{-2700}|=81+30\sqrt{-3}|$.

Here $A=81$, $B=30$, $q=3$; hence $\overline{A^2+B^2q}|^{\frac{1}{3}}=\sqrt[3]{81\times81+2700}|=21=p^2+l^2q=p^2+3l^2$. Now subtracting from 21, the Square of (p) ±3, which is a Divisor of A, there remains $12=3l^2$; or $l^2=4$, hence $l=2$, a Divisor of 30, and because A is positive, and the Factor p^2-3l^2q negative, therefore p must be (taken) negative, then will $A=p\times\overline{p^2-3l^2q}$ ($=-3\times\overline{9-36}$) $=81$, as it ought; for the same Reason $l=+2$; for then will $B=l\times\overline{3p^2-l^2q}=2\times\overline{27-12}=30$. So that the Root sought is $-3+2\sqrt{-3}|$.

EXAMPLE II.

What is the Cube Root of $2-\sqrt{-121}|=2-11\sqrt{-1}|$?

Here $A=2$, $B=-11$, $q=1$, therefore $\sqrt[3]{A^2+B^2q}|=\sqrt[3]{2\times2+121}|=\sqrt[3]{125}=5=p^2+l^2q$; and subtracting from 5, the Square of (p) ±2, a Divisor of A (2) there remains

remains $l^2q=1$, consequently $l=\pm 1$; by taking $p=+2$, we have $A=p\times\overline{p^2-3l^2q}=2$, and by taking $l=-1$, we have $B=l\times\overline{3p^2-l^2q}=-11$.

Therefore the Root required is $2-\sqrt{-1}$.

SCHOLIUM.

This Method will succeed when $\sqrt[3]{A^2+B^2q}$ is rational; and since $\sqrt[3]{A^2+B^2q}=p^2+l^2q$, p being ever a Divisor of A, its Value may therefore be found in a very few Trials, whence that of l will be readily known; and the Roots of Cubic Equations (whose Solutions would fail by the Theorems demonstrated on Page 178) may be obtained by Cardan's Form, when the Surd Cube Roots can be extracted as above; for then the irrational Parts of the Root will destroy one another: for

EXAMPLE.

Let there be given $x^3-15x=4$, to find x.

Put $3mn-15=0$, or $mn=5$; and write $m+n$ for x in the given Equation, and from the Equation thence arising, subtracting $3mn-15=0$, multiplied by $m+n$, there remains $m^3+n^3=4$, from the Square of which take 4 Times the Cube of the Equation $mn=5$, and you will have $m^6-2m^3n^3+n^6=-484$, hence $m^3-n^3=\sqrt{-484}$ $=22\sqrt{-1}$: This Equation added to, and subtracted from $m^3+n^3=4$, and the Cube Root of the Sum and Difference extracted, gives $m=\sqrt[3]{2+11\sqrt{-1}}=2+\sqrt{-1}$, and $n=\sqrt[3]{2-11\sqrt{-1}}=2-\sqrt{-1}$; and consequently x $(=m+n)=2+\sqrt{-1}+2-\sqrt{-1}=4$.

There is another Method of extracting the Roots of Surd Binomials: for an Instance whereof let it be proposed to extract the Cube Root of $-10+\sqrt{-243}$.

Put

Put $x \pm \sqrt{y}$ for the Root sought; then A being $=-10$, and $B=\sqrt{-243}$; we have $A^2-B^2=100+243=343=D=\overline{x^2-y}|^3$, hence $x^2-y=D^{\frac{1}{3}}=\sqrt[3]{343}=7$, or $y=x^2-D^{\frac{1}{3}}=x^2-7$; by putting the odd Terms in the Cube of $x+\sqrt{y}$, equal to A, we have $x^3+3xy=A=-10$, and by writing x^2-7 for its equal y, we have $x^3+3x^3-21x=A=-10$, or $4x^3-21x+10=0$: Here, when this Method succeeds, x must be a rational Number, and by trying the Divisors of the last Term 10, I find $x=2$; therefore $y=x^2-7=-3$, and consequently $x+\sqrt{y}=2+\sqrt{-3}$, the Root required.

In this Manner you may extract the n^{th} Roots of Binomials, but it is to be observed, that if D $(=A^2-B^2)$ is not a perfect n^{th} Power, then neither Term of the Root will be rational.

From this and the preceding Remarks, it will be easy to see at first, whether the Quantity proposed admits of such a Root as you would find; by which Means the Trouble of trying the whole Operation, when it will not succeed, may always be avoided.

Here follow promiscuous Problems for a further Illustration of the foregoing Rules.

PROBLEM I.

A Vintner would mix Wine at 10d. the Quart with another at 6d. to make a 100 Quarts, to be sold at 7d. per Quart; How much of each must be taken?

For the Numbers 10, 100, 6, and 7, put a, b, c, and d respectively; let x denote the Quantity at 10d. and y that at 6d. per Quart; then it will be $1 : a :: x\ a\ x$, Value of y Quarts:

Whence $\left\{ \begin{array}{l} ax+cy=bd \\ x+y=b \end{array} \right\}$ per Question: c Times the

second

ſecond Equation taken from the firſt, leaves $ax-cx=bd-bc$, hence $x=\frac{bd-bc}{a-c}=25$, and by taking a Times the ſecond Equation from the firſt, we have $cy-ay=bd-ab$, and $y=\frac{ab-bd}{a-c}=75$.

PROBLEM II.

A Diſtiller propoſes to mix foreign Brandy, ſtanding him in 8s. a Gallon with Britiſh Spirits of 3s. per Gallon, in ſuch Proportion that he may gain 30 per Cent. by ſelling out the Compound at 9 Shillings a Gallon: What is that Proportion?

Suppoſe that with x Gallons of Brandy he mixes y Gallons of Spirits, then the Brandy will ſtand him in $8x$ Shillings, and the Spirits $3y$ Shillings, and the Value of the whole Mixture will be $8x+3y$; but the Value of $x+y$ Gallons, at 9 Shillings per Gallon is $9x+9y$; therefore by laying out $8x+3y$, he gains $9x+9y-8x-3y$, or $x+6y$; and ſo we have $8x+3y : x+6y :: 100 : 30$, [by the Queſtion] hence $240x+90y=100x+600y$, or $140x=510y$, and $14x=51y$; from which it appears, that for every 51 Gallons of Brandy, there muſt be taken 14 Gallons of Spirits, for $x : y :: 51 : 14$.

PROBLEM III.

What Number is that which if added ſeverally to 3, 19, and 51, ſhall make them three Proportionals?

Put x for the Number ſought, then adding x to each given Number, they become $x+3$, $x+19$, $x+51$, proportionals by the Queſtion, whence $x+3 : x+19 :: x+19 : x+51$, therefore $\overline{x+3}\times\overline{x+51}=\overline{x+19}^2$, hence $x^2+54x+153=x^2+38x+361$, or $16x=208$, therefore $x=\frac{208}{16}=13$. And the three Proportionals are 16, 32, 64.

PROBLEM

PROBLEM IV.

What is the mean Proportional between 8 and 18.

Put $a=8$, $b=18$, and $x=$ the Number required; then it will be $a:x::x:b$, hence $x^2=ab$, and $x=\sqrt{ab}=12$: for $8:12::12:18$; or $a:\sqrt{ab}::\sqrt{ab}:b$.

PROBLEM V.

A Perſon draws a certain Quantity of Wine out of a full Veſſel that held 81 Gallons, and then recruiting the Veſſel with Water, takes a ſecond Draught of as much Wine and Water together as before he did of Wine, and ſo he goes on four Draughts one after another, always taking the ſame Quantity at a Draught, and then filling up the Veſſel with Water; at laſt there was found by Proof to remain 16 Gallons of pure Wine in the Veſſel: How much did he take at each Draught.

Put $a=81$, $b=16$, and $x=$ the Number of Gallons of pure Wine left in the Veſſel after the firſt Draught, then will $a-x$ denote the Quantity taken at every Draught; and ſince the Veſſel is filled up with Water after every Evacuation, it is evident that the Wine itſelf muſt be diminiſhed at each Draught in the Proportion of a to x; therefore $a:x::x:\frac{x^2}{a}$ the Wine left in the Veſſel after the ſecond Draught: then $a:x::\frac{x^2}{a}:\frac{x^3}{a^2}$ the Wine left after the third Draught, and $a:x::\frac{x^3}{a^2}:\frac{x^4}{a^3}$ the Wine left after the fourth Draught; hence per Queſtion $\frac{x^4}{a^3}=b$, or $x=\sqrt[4]{a^3b}$, and $a-\sqrt[4]{a^3b}=27$ Gallons, the Quantity taken at every Draught, conſequently the Quantities of Wine drawn out were 27, 18, 12 and 8 Gallons.

PROBLEM

PROBLEM VI.

There are two ſquare Pieces of Land, ſuch that if the Side of the leſs be augmented one Chain, and the Side of the greater diminiſhed three Chains, their Areas will be equal; but if the Side of the leſs Square be diminiſhed three Chains, and that of the greater increaſed three Chains, their Areas will be to each other as 2 to $24\frac{1}{2}$: Quere the Areas of thoſe Squares?

Put x for the Side of the leſs, and y for that of the greater Square, then per Queſtion $\overline{x+1}|^2 = \overline{y-3}|^2$, conſequently $x+1=y-3$, or $x+4=y$; again, per Queſtion, $\overline{x-3}|^2 : \overline{y+3}|^2 :: 2 : 24\frac{1}{2}$, or by writing $x+4$ for its equal y, it will be $\overline{x-3}|^2 : \overline{x+7}|^2 :: 2 : 24\frac{1}{2}$, hence $\overline{x-3}|^2 \times 24\frac{1}{2} = \overline{x+7}|^2 \times 2$, or $\overline{x-3}|^2 \times 49 = \overline{x+7}|^2 \times 4$, therefore $\overline{x-3} \times 7 = \overline{x+7} \times 2$, or $7x-21=2x+14$, whence $x=\frac{35}{5}=7$, and $y\ (=x+4)=11$, conſequently the Areas required are (x^2) 49 and (y^2) 121, Square Chains, equal to 4A. 3R. 24P. and 12A. 0R. 16P. reſpectively.

PROBLEM VII.

Divide 40 into two ſuch Parts that their Product may be equal to 256.

Put $a=20$, $b^2=256$; $a+x=$ the greater, and $a-x$ = the leſs Number required, then per Queſtion $\overline{a+x} \times \overline{a-x} = b^2$, or $a^2-x^2=b^2$; and $x=\sqrt{a^2-b^2}$, therefore $a+\sqrt{a^2-b^2}=32$, and $a-\sqrt{a^2-b^2}=8$, are the two Numbers ſought.

PROBLEM VIII.

A Man intends to travel as many Days as he has Crowns: It happens that every following Day of his Journey he had as many Crowns as he had the Day before, beſides

besides two Crowns over and above; and when he came to his Journey's End finds he had in all 45 Crowns: How many Crowns had he at first.

Put x for the Crowns he had when he set out, which, by the Question were equal in Number to the Days he travelled, and the first Term x is also equal to the Number of Terms in an Arith. Prog. whose common Difference is 2, whence by Case I. in Arith. Prog. we have $x+2x-2$, or $3x-2$, for the last Term; therefore $\overline{4x-2}$ is the Sum of the Extremes, and $\overline{4x-2} \times \frac{x}{2}$, or $2x^2-x$, is the Sum of all the Series, which, by the Question, is equal to 45 Crowns, hence $2x^2-x=45$, or $x^2-\frac{1}{2}x=22.5$, and $x=\frac{1}{4}+\sqrt{22.5625}=5$. For $5+7+9+11+13=45$.

PROBLEM IX,

A Man buys a Horse, which he sells again for 56 Crowns, and gains as many Crowns in the 100 as the Horse cost him: How much did he give for the Horse.

Put $a=100$, $b=56$, and $x=$ the Crowns that the Horse cost, then will $b-x$ be the Crowns which he gained by selling the Horse, and per Question $x : b-x :: a : x$, hence $x^2+ax=ab$, and $x=\sqrt{ab+\frac{a^2}{4}}-\frac{1}{2}a=40$.

PROBLEM XX.

A Man being asked what a Clock it was, answered, that it was between eight and nine, and that the Hour and Minute Hands were both together: Quere the precise Time when this happened?

The Minute Hand points to the Hour 12 at the End of every Hour, therefore it pointed to 12 at 8 o'Clock, at which Time the Hour Hand was $\frac{8}{12}$ of the Circumference of the Dial-Plate before the Minute Hand; but in an Hour the Minute Hand gains of the Hour Hand $\frac{11}{12}$ of the Circumference; for the Minute Hand goes the whole Round

Round in an Hour, the other goes only $\frac{1}{12}$ of it in that Time; this premised, put $a=11$, $b=8$, $c=1$ Hour; and $x=$ the Time required from 8 o'Clock to the next Conjunction of those Hands, then it will be $a : c :: b : x$, hence $ax=bc$, and $x=\frac{bc}{a}=\frac{8}{11}=43$ M. 38 Sec. $\frac{2}{11}$ past Eight.

Since the Value of b will ever increase or decrease with the Time from Twelve o'Clock, it follows that $\frac{bc}{a}$ will universally denote the Time of any Conjunction; thus in the next Conjunction b will $=9$, and then $\frac{bc}{a}$ will $=\frac{9}{11}=49$M. $\frac{1}{11}$ past Nine o'Clock; and when b becomes equal to 11, then a being likewise equal to 11, we shall have $\frac{bc}{a}=\frac{11}{11}=1$ Hour, this shews that the Minute and Hour Hands will always be in conjunction at Twelve o'Clock, which proves the Grounds of the Solution.

PROBLEM XI

Required the Dimensions of the greatest Cube that can be cut out of a Globe 12 Inches in Diameter.

Put x for the Side of the Cube, then $\sqrt{2x^2}$, or $x\sqrt{2}$ will be the Diagonal of its Side, and $x\sqrt{3}$ is the Diagonal passing through the Centre of the Cube, and is by the Nature of the Question equal to 12 Inches, the Diameter of the Globe, hence $x\sqrt{3}=12$, and $x=\frac{12}{\sqrt{3}}$ $=4\sqrt{3}$.

Hence it appears, that the Side of the Cube will always be equal to $\frac{1}{3}$ of the Diameter of the Globe multiplied into $\sqrt{3}$.

PROBLEM

PROBLEM XII.

There is a Cylindrical Tub, whose Diameter is 30 Inches, and Depth 60,5 Inches: I demand the Diameter of another Cylindrical Tub that shall contain the same Quantity, when its Depth is but 50 Inches?

Put $a=60,5$, $b=30$, $c=50$, $d=,7854$; and $x=$ the Diameter required, then the Solidities of the Cylinders will be cdx^2 and ab^2d respectively, hence per Question we have

$cdx^2=ab^2d$, or $cx^2=ab^2$, and $x=b\sqrt{\frac{a}{c}}=33$ Inches.

PROBLEM XIII.

Suppose the Top Diameter of a Conical Tub, be 54 Inches, and its Depth 36 Inches, what must be the internal Length of its Side and its Bottom Diameter, when it will hold 245 Ale Gallons.

Put $s=69090$ the solid Inches in 245 Ale Gallons, $a=54$, $c=36\div3=12$, $n=,7853982$, and $x=$ the Diameter sought; then by a well known Theorem, we have

$\overline{x^2+ax+a^2}\times cn=s$, or $x^2+ax=\frac{s}{cn}-a^2$; hence $x=$

$\sqrt{\frac{s}{cn}-\frac{3}{4}a^2}-\frac{1}{2}a=44,72$ Inches; and the Square Root of $\overline{36}|^2+\overline{4,64}|^2$ the Sum of the Squares of the Depth, and of Half the Difference of the Diameters, gives 36,3 Inches nearly, for the internal Length of the Side as required.

PROBLEM XIV.

If a Globe of Metal, five Inches in Diameter weighs 25 pound, what will be the Diameter of another Globe, composed of the same species of Metal, that weighs 216 pound?

As

As the Weight of one Globe is to the Cube of its Diameter, so is the Weight of another Globe to the Cube of its Diameter.

Therefore putting $a=27$, $b=5$, $c=216$; and $x=$ the Diameter required, we have $a:b^3::c:x^3$, or $ax^3=b^3c$, and $x=\sqrt[3]{\frac{b^3c}{a}}=b\sqrt[3]{\frac{c}{a}}=10$ Inches. Hence it will be universally $b^3:a::\frac{b^3c}{a}:c$, or in Numbers $5^3:27$lb. $::\overline{10}|^3:216$lb.

So that by having the Diameters of two Globes, and the Weight of one of them, the Weight of the other is found by Proportion.

For as the Cube of the Diameter of one Globe is to its Weight, so is the Cube of the Diameter of the other Globe to its Weight.

PROBLEM XV.

A Person has a Rectangular Cistern 6 Feet long, 5 wide, and 3 deep; but wants a new one that will hold 3,375 Times as much, and have the Dimensions in the same Proportion to one another as the old one: Quere, those Dimensions.

Put $a=6$, $b=5$, $c=3$, $d=3{,}375$; and $x=$ the Length sought.

Then abc is the Solidity of the old Cistern, and $abcd$ is that of the new one, whence it will be $a^3:abc::x^3:abcd$, therefore $abcx^3=a^4bcd$, or $x^3=a^3d$, and $x=a\sqrt[3]{d}=9$ Feet the Length, then $a:a\sqrt[3]{d}::b:b\sqrt[3]{d}=7{,}5$ Feet the Breadth; and $a:a\sqrt[3]{d}::c:c\sqrt[3]{d}=4{,}5$ Feet the Depth.

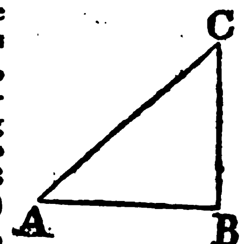
Hence this Rule. Multiply the Cube Root of the Number of Times that the proposed Cistern is to be greater or less than the given one, by the given Dimen-

S sions

fions ſeverally, and the reſpective Products will be the Dimenſions required.

PROBLEM XVI.

Given the Baſe (equal to 200 Feet) of an inclined Plane, on which a Body of 60 Pounds is ſuſtained by a Power of 36 Pounds acting in a Direction perpendicular to the Horizon: Quere the Height and Length of the Plane.



In the Right-angled Triangle A B C, let A C denote the Length and B C the Height of the Plane, then by a known Principle in Mechanics, the force of a Body acting on the inclined Plane A C, is to a Force acting in the Direction CB (perpendicular to the Horizon AB) as B C the Height of the Plane, is to its Length AC; this premiſed, put $a=60$, $c=36$, AB$=b=200$; and B C$=x$; then it will be $c:a::x:\frac{ax}{c}=$ A C; hence by Euc. 47.1. we have $\frac{a^2x^2}{c^2}-x^2=b^2$, or $a^2x^2-c^2x^2=b^2c^2$, whence $x=\frac{bc}{\sqrt{a^2-c^2}}=150$ Feet, and $\frac{ax}{c}$, or A C$=\frac{ab}{\sqrt{a^2-c^2}}=250$ Feet.

PROBLEM XVII.

The Difference between the internal Length and Breadth of a certain Rectangular Cooler, is equal to 5 Times its Depth, and its Breadth is to its Depth as 27 to 4: Quere its Dimenſions when it contains 486 Ale Gallons.

Put $s=137052$, the Cubic Inches in 486 Ale Gallons, $m=27$, $n=4$; $x=$ the Length, and $y=$ the Breadth of the

Cooler; then will $\frac{s}{xy}=\frac{ny}{m}$ the Depth, and $x-y=\frac{5s}{xy}$, per Question, therefore $x-y=\frac{5ny}{m}$ hence $y=\frac{mx}{m+5n}$: put $m+5n\ (=47)=c$, and write $\frac{mx}{c}$ for y in the Equation $x-y=\frac{5s}{xy}$, and you will have $x-\frac{mx}{c}=\frac{5cs}{mx^2}$, hence $x=\sqrt[3]{\frac{5c^2s}{cm-m^2}}=141$, consequently, $y=\frac{mx}{c}=81$, and $\frac{ny}{m}=12$ Inches the Depth.

PROBLEM XVIII.

A and B are two Artificers, A is the most expert Workman, so that the Cube of the Work performed by A is equal to 6 Times the Square of what B can do of it in the same Time: Now if A and B do a Piece of Work together, how much of it will B perform?

Let the whole Work be denoted by Unity, and put x for the Work performed by A, then will $1-x$ be the Work done by B; whence per Question $x^3=6\times\overline{1-x}|^2$, or $x^3=6-12x+6x^2$, therefore $x^3-6x^2+12x=6$, here completing the Cube, and extracting the Root, we have $x-2=\sqrt[3]{-2}$, hence $x=2+\sqrt[3]{-2}=0.74007895+$, $=$ A's Part of the Work, and $1-x=0.25992105=$B's Part.

Or putting $1-x$ for A's Part, and x for B's, we have $\overline{1-x}|^3=6x^2$, or $x^3+3x^2+3x=1$; here by adding 1 to both Sides of the Equation, and extracting the Cube Root, we get $x=\sqrt[3]{2}-1=0.259921+$, $=$ B's Part of the Work, and $(1-x=)\ 2-\sqrt[3]{2}=0.74007895=$A's Part, as before.

 PROBLEM

PROBLEM XIX.

The Sum of the Diagonal and Perimeter of a Rectangled Parallelogram is 880; Quere, its Area, when its Length is a mean proportional between its Breadth and Diagonal.

Put x for the Breadth and y for the Length of the Parallelogram, then per E. 1. 47. $\sqrt{x^2+y^2}$ is the Diagonal, and per Question,

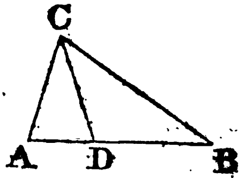
$$2x+2y+\sqrt{x^2+y^2}=880=s, \text{ and}$$
$$x\sqrt{x^2+y^2}=y^2:$$

From x Times the first Equation take the second, and you will have $2x^2+2xy=sx-y^2$, and, by squaring the second Equation $x^4+x^2y^2=y^4$: Here, by completing the Square $x^4+x^2y^2+\frac{1}{4}y^4=\frac{5}{4}y^4$; hence $x^2+\frac{1}{2}y^2=\frac{1}{2}\sqrt{5}\times y^2$, or $x^2=\overline{\frac{1}{2}\sqrt{5}-\frac{1}{2}}\times y^2$; therefore $x=\sqrt{\frac{1}{2}\sqrt{5}-\frac{1}{2}}\times y$: Put $a=\sqrt{\frac{1}{2}\sqrt{5}-\frac{1}{2}}$ $(=.78615)$, then will $x=ay$, and by writing ay for x in the third Equation, there will arise $2a^2y^2+2ay^2=asy-y^2$, hence $y=\frac{as}{2a^2+2a+1}=$ 181.65597, the Length, and $x\ (=ay)=\frac{a^2s}{2a^2+2a+1}=$ 142.80884, the Breadth, and consequently $xy=\frac{a^3s^2}{\overline{2a^2+2a+1}|^2}=25942$, the Area required.

PROBLEM XX.

In a Plane Triangle ABC, is given CD bisecting the Vertical Angle and terminating at the base $=30$, and the Product of each Segment of the Base into its adjacent or next Side $=486$ and 2400, to find the Segments of the Base and the Sides.

Put



Put $a=486$, $b=2400$, $c=30$; AD $=x$, and BD $=y$, then will AC $=\frac{a}{x}$, BC $=\frac{b}{y}$, and $xy+c^2=\frac{ab}{xy}$, or $x^2y^2+c^2xy=ab$; here by completing the Square, &c. we have $xy=\sqrt{ab+\frac{1}{4}c^4}-\frac{1}{2}c^2=720=d$, hence y, or DB $=\frac{d}{x}$, therefore BC $(=b\div\frac{d}{x})=\frac{bx}{d}$: And AD : AC :: DB : BC, that is, $x:\frac{a}{x}::\frac{d}{x}:\frac{bx}{d}$, whence $bx^4=ad^2$, and $x=\sqrt[4]{\frac{ad^2}{b}}=18$, hence $y=40$, AB $=58$, BC $=60$, and AC $=27$.

PROBLEM XXI.

Find three Numbers or Lines in Arithmetical Progression, whose Product shall be 35, and their common Difference 6.

Put $x+6$, x and $x-6$ for the three required Numbers, then per Question, we have $\overline{x+6}\times x\times\overline{x-6}=35$, or $x^3-36x=35$, therefore $x^3-36x-35=0$, this Equation divided by $x+1$, gives $x^2-x-35=0$, and $x=\sqrt{35,25}+,5$, hence the Numbers sought are $\sqrt{35,25}+6,5$, $\sqrt{35,25}+,5$, and $\sqrt{35,25}-5,5$.

PROBLEM XXII.

Find three Numbers whose Sum shall be 2000, moreover if the Cube Root of the third, and Square Root of the first be subtracted from the second, there shall remain 362 and 341 respectively.

 Put

Put x^2, y and z^3 for the Numbers sought, then per Question $x^2+y+z^3=2000$,

$$y-z = 362,$$
$$\text{and } y-x = 341,$$

Taking the third Equation from the second we have $x=z+21$, and by the second Equation $y=z+362$; these Values of x and y being substituted in the first Equation, there arises $z^3+z^2+43z-1197=0$. Here, trying $(1, 3, 9)$ Divisors of the last Term (1197) I find $z=9$, consequently the Numbers are 900, 371 and 729.

PROBLEM XXIII.

Find four Numbers, such, that if to the Biquadrate of each of them nine Times its first, and ten Times its second Power be added; and from that Sum, seven Times its Cube subtracted, the Remainder shall be nine.

Put x for either of the required Numbers, then per Question we have

$$x^4+10x^2+9x-7x^3=9, \text{ or}$$
$$x^4-7x^3+10x^2+9x-9=0,$$ an Equation answering to the first Case demonstrated (on Page 198) for Biquadratics; where $A=0$, $a=-7$, $b=10$, $c=9$, $d=-9$; $C=\sqrt{-a}=3$, and $B=\frac{-c}{2C}=-1{,}5$: These Numbers being written respectively for a, B and C, in the second general Theorem for Biquadratics, we have $x=\pm\overline{0{,}75+1{,}75\pm\sqrt{3{,}625\pm2{,}625\pm3}}$, four Numbers, namely 3, -1; and $2{,}5\pm\sqrt{3{,}25}$, each of which will solve the Problem, and these are the only Numbers that can answer the Question in the Case proposed.

This may be likewise solved from the original Equation by the Method explained, at Page 205.

Thus, by adding $2{,}25x^2-9x$ to both Sides of the Equation $x^4-7x^3+10x+9x=9$, we have $x^4-7x^3+12{,}25x^2=2{,}25x^2-9x+9$; or $x^2-3{,}5x=1{,}5x-3$, hence $x^2-5x+3=0$, and dividing the Equation $x^4-7x^3+10x^2+9x-9=0$, by x^2-5x+3, the Quotient is $x^2-2x-3=0$;

hence

hence $x=1\pm\sqrt{4}=1\pm2$; and from the Equation $x^2-5x+3=0$, we get $x=2,5\pm\sqrt{3,25}$ the other two Roots, as before.

PROBLEM XXIV.

Find two Numbers whose Sum being added to the Sum of their Squares shall make 62; and their Sum being added to their Product shall be 34.

Put x and y for the Numbers sought, then $x^2+y^2+x+y=62$, and $xy+x+y=34$; adding twice the second Equation to the first, we have $\overline{x+y}|^2+3\times\overline{x+y}=130$, hence $x+y+1,5=\sqrt{132,25}=11,5$, and $y=10-x$; this substituted for y in the second Equation, gives $10x-x^2+10=34$, or $x^2-10x=-24$: Hence $x=5\pm1$, and these two Numbers 6 and 4 will solve the Question.

PROBLEM XXV.

The Area of a Rectangular Garden, and the Diagonal in one Sum is 73 Perches, and the Difference of its Sides is 7 Perches: Quere its Dimensions.

Put x for the less Side, then will $x+7$ be the greater, and $\sqrt{2x^2+14x+49}$ is the Diagonal, hence per Question, we have $\sqrt{2x^2+14x+49}+x^2+7x=73$, or $\sqrt{2x^2+14x+49}=73-x^2-7x$, hence by squaring both Sides, and, by Transposition, we have $x^4+14x^3-99x^2-1036x=-5280$; this Equation is one of the Class that may be unfolded by Division, as has been demonstrated, on Page 202, whence we have $\overline{x^2+7x}|^2-148\times\overline{x^2+7x}=-5280$; or $x^2+7x-74=\pm14$, hence $x^2+7x=60$, and $x=\sqrt{72,25}-3,5=5$ Perches, consequently the Length is 12 Perches, the Diagonal 13; and the Area is 60 Square Perches.

Otherwise, Put $73=s$, $7=2d$, $x+d=$ the Length of the Garden, then $x-d$ will be its Breadth, and $\sqrt{2x^2+2d^2}$ is its Diagonal, hence $\sqrt{2x^2+2d^2}+x^2-d^2=s$, or

$\sqrt{2x^2+2d^2} + x^2 = d^2 + s = 85,25 = m$, therefore $\sqrt{2x^2+2d^2} = m - x^2$, or $2x^2+2d^2 = m^2 - 2mx^2 + x^4$, this solved gives $x = \sqrt{m+1-\sqrt{2d^2+2m+1}} = 8,5$, hence the Dimensions are become known, as above.

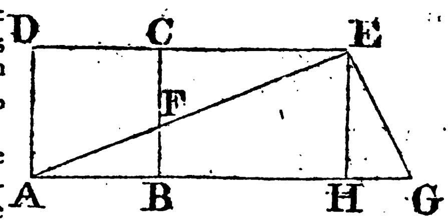
PROBLEM XXVI.

Given the continual Product of the four Sides, in Arithmetical Progression, of a Trapezium equal to 176985 (p), and their common Difference 4 (d); to find the Sides.

Put x for the less Side, then per Question, we have $x \times \overline{x+d} \times \overline{x+2d} \times \overline{x+3d} = p$, that is $x^4 + 6dx^3 + 11d^2x^2 + 6d^3x = p$, or $\overline{x^2+3dx}^2 + 2d^2 \times \overline{x^2+3dx} = p$, hence $x^2 + 3dx + d^2 = \sqrt{d^4+p}$, and $x = \sqrt{1,25d^2 + \sqrt{d^4+p}} - 1,5d = 15$; consequently the other three Sides are 19, 23 and 27.

PROBLEM XXVII.

Having the Side of a Square ABCD; it is required to draw a Line from the Angle A as AE to cut the Side BC in F, and meet DC (produced beyond C) in E, so that the Part EF may be of a given Length.



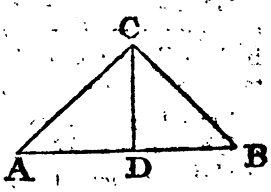
Let $AB = 4 = a$, $EF = 6 = b$; and $DE = x$, then will $CE = x - a$, and $AE = \sqrt{x^2+a^2}$, hence by similar Triangles, we have $DE : AE :: CE : EF$, that is, $x : \sqrt{x^2+a^2} :: x-a : b$.

Therefore $bx = \overline{x-a} \times \sqrt{x^2+a^2}$, hence $b^2x^2 = x^4 - 2ax^3 + 2a^2x^2 - 2a^3x + a^4$, or $x^4 - 2ax^3 + \overline{2a^2-b^2} \times x^2 - 2a^3x + a^4 = 0$: Here, by adding $\overline{a^2+b^2} \times x^2$ to both Sides, and

and extracting the Root, we have $x^2 - ax + a^2 = \sqrt{a^2+b^2} \times x$, or $x^2 - \overline{a + \sqrt{a^2+b^2}} \times x = -a^2$; put $2c = a + \sqrt{a^2+b^2}$ then $x^2 - 2cx = -a^2$, and $x = \sqrt{c^2 - a^2} + c = 9,532674$, hence CE $= 5,532674$; otherwise make EG perpendicular to AE, then will AF $=$ EG, put BG $= y$, and AF $=$ EG $= z$, then will AG $= y + a$, and AE $= z + b$; and as AB : AF :: AE : AG, that is, $a : z :: z + b : y + a$, hence $ay + a^2 = z^2 + bz$, and $\overline{y+a}^2 = \overline{z+b}^2 + z^2$ (per Euclid, 47. 1.) from this Equation take twice $ay + a^2 = z^2 + bz$, and you will have $y^2 + a^2 = b^2$, and $y = \sqrt{a^2+b^2}$, this substituted for y, in the Equation $ay + a^2 = z^2 + bz$, gives $z^2 + bz = a\sqrt{a^2+b^2} + a^2$; and $z = \sqrt{a\sqrt{a^2+b^2} + a^2 + \frac{1}{4}b^2} - \frac{1}{2}b$, consequently AE $= \sqrt{a\sqrt{a^2+b^2} + a^2 + \frac{1}{4}b^2} + \frac{1}{2}b$; hence the Point E, or the Distance DE may be readily found.

PROBLEM XXVIII.

In the Right-Angled Triangle ABC, there is given the Sum of the Legs $= 70$, and the Sum of the Hypothenuse and Perpendicular falling from the Right-Angle thereon $= 74$; to determine the Triangle.



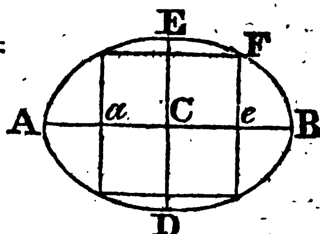
Put AC + BC $= 70 = a$, AB + CD $= 74 = b$; AC $= x$, and AB $= y$, then will BC $= a - x$, and CD $= b - y$; hence per Euc. 47, 1. we have AC^2 + BC^2 = AB^2 $= x^2 + \overline{a-x}^2 = y^2$, or $y = \sqrt{x^2 + \overline{a-x}^2}$ and as AB : BC :: AC : CD, or $\sqrt{x^2 + \overline{a-x}^2} : a - x :: x : b - \sqrt{x^2 + \overline{a-x}^2}$ hence $b\sqrt{x^2 + \overline{a-x}^2} = x^2 - ax + a^2$; therefore $2b^2x^2 + a^2b^2 - 2ab^2x = x^4 - 2ax^3 + 3a^2x^2 - a^3x + a^4$, or $x^4 - 2ax^3 + \overline{3a^2 - 2b^2} \times x^2 + \overline{2ab^2 - 2a^3} \times x - a^2b^2 + a^4 = 0$; hence, $\overline{x^2 - ax}^2 + \overline{2a^2 - 2b^2} \times \overline{x^2 - ax} = a^2b^2 - a^4$; here by completing the Square, &c. we have $x^2 - ax + a^2 - b^2 = -b\sqrt{b^2 - a^2}$, and

$x =$

$x=\sqrt{b^2-\frac{1}{4}a^2-b\sqrt{b^2-a^2}}+\frac{1}{2}a=40$: Hence y, or AB $=\sqrt{x^2+\overline{a-x}|^2}=50$, BC $=30$, and CD $=24$.

PROBLEM XXIX.

Given the Sum of the Axis 210, and Side 72 of the inscribed Square of the Ellipse ABDE; to find the Axis AB and DE separately.



Put $s=210\div2=105$, $d=a\mathrm{C}=\mathrm{C}e=e\mathrm{F}=36$; AC $=$ BC $=x$, then will $\mathrm{A}e=x+d$, $\mathrm{B}e=x-d$, CE $=s-x$, and by the Property of the Ellipse, it will be $\overline{x+d}\times\overline{x-d}:d^2::x^2:\overline{s-x}|^2$, hence $x^4-2sx^3+\overline{s^2-d^2}\times x^2+2d^2sx-d^2s^2=d^2x^2$, or $\overline{x^2-sx}|^2-2d^2\times\overline{x^2-sx}=d^2s^2$ therefore $x^2-sx-d^2=\pm d\sqrt{d^2+s^2}$, and $x=\sqrt{d^2+\frac{1}{4}s^2-d\sqrt{d^2+s^2}}+\frac{1}{2}s=60$, consequently AB $=2\sqrt{d^2+\frac{1}{4}s-d\sqrt{d^2+s^2}}+s=120$, and DE $=s-2\sqrt{d^2+\frac{1}{4}s^2-d\sqrt{d^2+s^2}}=90$.

PROBLEM XXX.

Find six Numbers in Geometrical Progression, whose Sum shall be 315 (a), and the Sum of the two Extremes 165 (s).

Put x for the first Term, and r for the Ratio, then per Question $x+rx+r^2x+r^3x+r^4x+r^5x$, or $x\times\frac{r^6-1}{r-1}=a$, and $x\times\overline{r^5+1}=s$, or $x=\frac{s}{r^5+1}$, this being written for

x in the first Equation, we have $\frac{s}{r^5+1}\times\frac{r^6-1}{r-1}=a$, or $s\times\overline{r^6-1}=a\times\overline{r^5+1}\times\overline{r-1}$: here, dividing by r^2-1, we get $s\times\overline{r^4+r^2+1}=a\times\overline{r^4-r^3+r^2-r+1}$, hence $\overline{a-s}\times r^4-ar^3+\overline{a-s}\times r^2-ar+a-s=0$, or $r^4-\frac{a}{a-s}r^3+r^2-\frac{a}{a-s}r+1=0$; put $4c=\frac{a}{a-s}$, then $r^4-4cr^3+r^2-4cr+1=0$; here, by adding $\overline{4c^2+1}\times r^2$ to both Sides, and extracting the Square Root, we have $r^2-2cr+1=2\sqrt{c^2+\frac{1}{4}}\times r$, or $r^2-\overline{2\sqrt{c^2+\frac{1}{4}}+2c}\times r=-1$; and $r=\sqrt{2c\sqrt{c^2+\frac{1}{4}}+2c^2-\frac{3}{4}}+\sqrt{c^2+\frac{1}{4}}+c=2$; hence x $(=\frac{s}{r^5+1})=5$, consequently 5, 10, 20, 40, 80, 160, are the six Numbers required.

PROBLEM XXXI.

Two Persons A and B having an equal Claim to an Annuity of 100l. to continue for 30 Years, agree to share it between them in this Manner, namely, A for his Part is to enjoy the whole Annuity for the first ten Years; B and his Heirs being to have the entire Reversion thereof for the remaining 20 Years: Quere the Rate of compound Interest allowed in this Contract, and the present Worth of the Annuity corresponding?

Put x for the Ratio, or the Amount of 1l. in one Year, then by Geometrical Progression, (Page 128) we have $100+100x+100x^2+100x^3+100x^4$, &c. continued to 30 Terms, equal to $\frac{100x^{30}-100}{x-1}$ the Amount of (100l.) the Annuity in 30 Years; and one Pound ready Money is equivalent to x^{30} (the Amount of one Pound) to be received

ceived 30 Years hence, therefore $x^{30} : 1 :: \frac{100x^{30}-100}{x-1} :$ $\frac{100x^{30}-100}{x^{30}\times \overline{x-1}}$ the present Worth of the Annuity for 30 Years, and $\frac{100x^{10}-100}{x^{10}\times \overline{x-1}}$ is its present Worth for 10 Years, which must, by the Nature of the Question, be equal to half the present Worth for 30 Years; hence $\frac{100x^{10}-100}{x^{10}\times \overline{x-1}} = \frac{50x^{30}-50}{x^{30}\times \overline{x-1}}$, or $\frac{2x^{10}-2}{x^{10}} = \frac{x^{30}-1}{x^{30}}$ therefore $2x^{30}-2x^{20}=x^{30}-1$, or $x^{30}-2x^{20}+1=0$; here $x=1$, which Root will not solve the Problem, but since one of the Values of x is 1, it is evident that $x^{10}-1=0$, therefore dividing $x^{30}-2x^{20}+1=0$, by $x^{10}-1$, we have $x^{20}-x^{10}-1=0$, or $x^{20}-x^{10}=1$; and by completing the Square, &c. $x^{10}-,5=\sqrt{1,25}=1,1180339887$, or $x^{10}=1,6180339887$, hence $x^5=1,27201964$. Here I find $x=1,05$ nearly, put $r=1,049$, $s=1,27201964$, and write $z+r$ for x, then taking only the first Power of z, we have $5r^4z+r^5=s$, and $z=\frac{s-r^5}{5r^4}=,0002979$, this added to 1,049, gives $x=1,0492979$, consequently the Rate of Interest is 4,92979, or 4l. 18s. 7d. per Cent. per Annum, and $\frac{100x^{30}-100}{x^{30}\times \overline{x-1}}$, or its equal $\frac{200x^{10}-200}{x^{10}\times \overline{x-1}} = \frac{123,60679774}{,0797656} = 1549,625 = 1549$l. 12s. 6d. the present Worth of the Annuity, as required.

PROBLEM XXXII.

Given the Sum 42 (s) of the Legs of a Right-Angled Triangle, and their Difference multiplied by the Hypothenuse equal to 180 (d); to find the Legs.

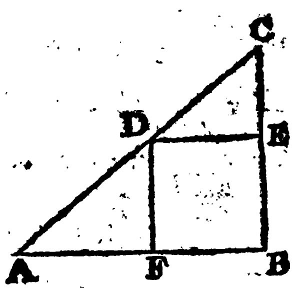
Put

Put x for the less Leg, then $s-x$ will be the greater, their Difference is $s-2x$; and the Hypothenuse is $\sqrt{2x^2-2sx+s^2}$; hence per Question, we have $\sqrt{2x^2-2sx+s^2}\times\overline{s-x}=d$, or $8x^4-16sx^3+14s^2x^2-6s^3x+s^4=d^2$, therefore $x^4-2sx^3+1{,}75s^2x^2-\frac{3}{4}s^3x=\frac{1}{8}d^2-\frac{1}{8}s^4$, or $\overline{x^2-sx}|^2+\frac{3}{4}s^2\times\overline{x^2-sx}=\frac{1}{8}d^2-\frac{1}{8}s^4$; and $x^2-sx+\frac{3}{8}s^2=\frac{1}{2}\sqrt{\frac{1}{2}d^2+\frac{1}{16}s^4}$; and consequently $x=\frac{1}{2}s\pm\sqrt{\frac{1}{2}\sqrt{\frac{1}{2}d^2+\frac{1}{16}s^4}-\frac{1}{8}s^2}=18$, and 24 the Legs required.

Otherwise, by putting s for (21) half the Sum, and x for half the Difference of the Legs, we have $x+s$ for the greater and $s-x$ for the less Leg, and $\sqrt{2x^2+2s^2}\times 2x=d$ (per Question) hence $8x^4+8s^2x^2=d^2$, or $x^4+s^2x^2=\frac{1}{8}d^2$; and $x=\sqrt{\frac{1}{2}\sqrt{\frac{1}{2}d^2+s^4}-\frac{1}{2}s^2}$ therefore $s\pm\sqrt{\frac{1}{2}\sqrt{\frac{1}{2}d^2+s^4}-\frac{1}{2}s^2}=21\pm 3$, the two Legs as before.

PROBLEM XXXIII.

The Hypothenuse AC (70), of a Right-Angled Triangle ABC, and the Side DE (24) of the inscribed Square BEDF, being given, to find the other two Sides of the Triangle.

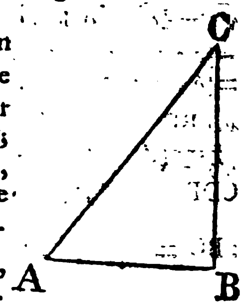


Put $a=70$, $b=24$; and CE $=x$, then will BC $=x+b$, and CE : DE :: CB : AB, that is $x:b::x+b:\frac{bx+b^2}{x}=$ AB; hence per Euc. 47.1, we have $\overline{x+b}|^2+\overline{\frac{bx+b^2}{x}}|^2=a^2$ or $x^4+2bx^3+2b^2x^2+2b^3x+b^4=a^2x^2$; here by adding b^2x^2 to both Sides, and extracting the Root, we have $x^2+bx+b^2=\sqrt{a^2+b^2}\times x$, or $x^2+\overline{b-\sqrt{a^2+b^2}}\times x=-b^2$; this solved,

solved, by completing the Square, &c. gives $x = \frac{1}{2}\sqrt{a^2+b^2} \pm \sqrt{\frac{1}{4}a^2 - \frac{1}{2}b\sqrt{a^2+b^2} - \frac{1}{2}b^2} - \frac{1}{2}b$; therefore $\frac{1}{2}\sqrt{a^2+b^2} + \frac{1}{2}b \pm \sqrt{\frac{1}{4}a^2 - \frac{1}{2}b\sqrt{a^2+b^2} - \frac{1}{2}b^2} = 49 \pm 7$, are the two required Sides of the Triangle.

PROBLEM XXXIV.

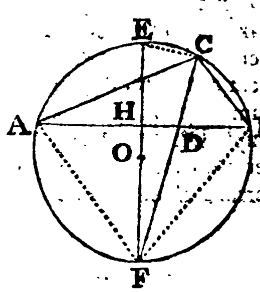
Having one Leg AB of a Right-Angled Triangle ABC, to find the other Leg BC, so that the Rectangle under their Difference (BC−AB) and the Hypothenuse AC, may be equal to the Area of the Triangle.



Put AB $=a$; and BC $=x$, then will AC $=\sqrt{x^2+a^2}$, and by the Question $\overline{x-a} \times \sqrt{x^2+a^2} = \frac{1}{2}ax$, or $x^4 - 2ax^3 + 2a^2x^2 - 2a^3x + a^4 = \frac{1}{4}a^2x^2$; here by adding a^2x^2 to both Sides, and extracting the Root, we have $x^2 - ax + a^2 = \frac{1}{2}a\sqrt{5} \times x$, or $x^2 - \overline{a + \frac{1}{2}a\sqrt{5}} \times x = -a^2$, this solved, by completing the Square, &c. gives $x = \frac{1}{2}a \times \overline{1 + \frac{1}{2}\sqrt{5} + \sqrt{\sqrt{5} - 1{,}75}}$.

PROBLEM XXXV.

Given the Base of a Triangle, the Line that bisects the Vertical Angle, and the Diameter of the circumscribing Circle to find the Sides.



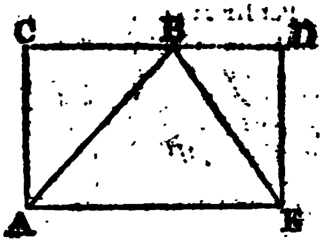
Draw the Diameter EF at Right-Angles to the Base AB, then will AH=HB, produce CD to F, and join CE; then the Triangle FCE being in the Semicircle FBE is Right-Angled at C, and by construction the Triangle FHD is Right-Angled at H; therefore putting AB $=b=194$, CD $=d=66$, EO $=$ FO $=r=100$; HD

$=x$, FD$=y$; we have AD$=\frac{1}{2}b+x$, BD$=\frac{1}{2}b-x$, and FH $=\sqrt{y^2-x^2}$; but (by Euc. 35.3) AD$\times$BD$=$CD$\times$FD, that is, $\overline{\frac{1}{2}b+x}\times\overline{\frac{1}{2}b-x}=dy$, hence $x^2=\frac{1}{4}b^2-dy$, now by writing $\frac{1}{4}b^2-dy$ for x in $\sqrt{y^2-x^2}=$FH, we have FH$=$ $\sqrt{y^2+dy-\frac{1}{4}b^2}$; and by Reason of the similar Triangles FEC, FHD, it will be FD :: FH :: FE : FC, or $y : \sqrt{y^2+dy-\frac{1}{4}b^2} :: 2r : y+d$, hence we have $y^2+dy=2r\sqrt{y^2+dy-\frac{1}{4}b^2}$, or $\overline{y^2+dy}^2=4r^2\times\overline{y^2+dy}-r^2b^2$, therefore $\overline{y^2+dy}^2-4r^2\times\overline{y^2+dy}=-r^2b^2$; or $y^2+dy-2r^2=2r\sqrt{r^2-\frac{1}{4}b^2}$, consequently $y=\sqrt{\frac{1}{4}d^2+2r^2+2r\sqrt{r^2-\frac{1}{4}b^2}}-\frac{1}{2}d=128,09345$, and $x=\sqrt{\frac{1}{4}b^2-dy}=30,9$; hence AD $=\frac{1}{2}b+x=127,9=m$, BD$=\frac{1}{2}b-x=66,1=n$; join AF and BF, then since FH$=\sqrt{y^2-x^2}$, we have AF$=$BF$=$ $\sqrt{y^2-x^2+\frac{1}{4}b^2}=157,6772=a$; and the Triangles ADF, CDB being similar, we have AD (m) : AF (a) :: CD (d) : BC $=\frac{ad}{m}=81,36587$, and BD (n) : BC $\left(\frac{ad}{m}\right)$ :: AD (m) : AC$=\frac{ad}{n}=157,43865$.

PROBLEM XXXVI.

A Ship sails from the Port A in Latitude 49° 12′ North, at the same Time another Ship sails from the Port E bearing due East from the first Port, on a Rhumb which makes with the Meridian an Angle equal to the Complement of the first Ship's Course; after some Time they meet each other at B in Latitude 50° North, and find the Sum of their Distances sailed equal to 140 Miles, the first Ship's Distance being the greatest: Required the Distance between the Ports, likewise the Course, Distance, and Departure of each Ship.

In



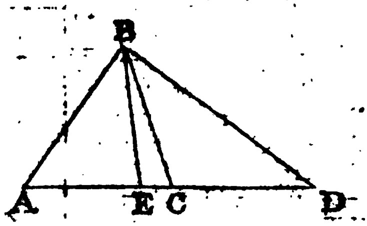
In the Right-Angled Parallelogram ACDE, let AC and ED represent the respective Meridians of the Ports A and E. Put $d=AC=ED=48$ Miles the Difference of Latitude, $AB+EB=140=s$; $AB=x$ the Distance sailed by the first Ship; then will $EB=s-x=$ the second Ship's Distance. Let v denote the natural Sine to the Radius 1. of the Angle ABC, which is equal to the Angle BED by the Question; then by Trigonometry, we have $v:d::1:x$, or $v=\frac{d}{x}$ and $1:s-x::v:sv-vx=\sqrt{\overline{s-x}|^2-d^2}$ $=BD$, per Euc. 47. 1. By writing $\frac{d}{x}$ for v in this Equation, and squaring both Sides, we have $x^4-2sx^3+\overline{s^2-2d^2}$ $\times x^2+2d^2sx=d^2s^2$, or $\overline{x^2-sx}|^2-2d^2\times\overline{x^2-sx}=d^2s^2$; hence $x^2-sx-d^2=\mp d\sqrt{d^2+s^2}|$, and $x=$ $\sqrt{d^2+\frac{1}{4}s^2-d\sqrt{d^2+s^2}}|+\frac{1}{2}s=80$ Miles; hence $s-x$, or $BE=60$, and $v=\frac{d}{x}=,6$ the natural Sine of 36° 52′ the second Ship's Course, corresponding to N. W. by N. ¾ W. consequently the first Ship's Course is N. E. ¼ E.

The Departures are $sv-vx=36=BD$, and (60 : 48 :: 80 :) $64=CB$; their Sum (100 Miles) is equal to the Distance between the Ports.

PROBLEM XXXVII.

In a Right-Angled Triangle ABD is givnn the Line BE bisecting the Right-Angle B and meeting the Hypothenuse AD in $E=24{,}244$, and the Line BC bisecting the Hypothenuse $=25$; to find the Sides and Area of the Triangle.

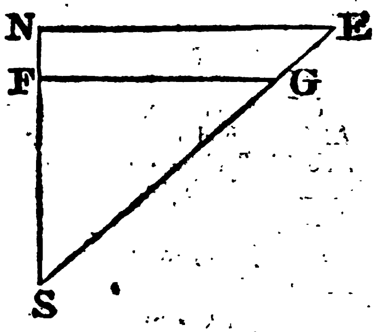
Put



Put BE $=24,244=a$, 2 AC, or AD $=50=b$, $s=,7071068$, the natural Sine of $45°$, the Angle ABE $(=\angle$ DBE) to the Radius 1; and AE $=x$; then by Trigonometry it will be $x:s::a:\frac{as}{x}=$ the Sine of the Angle A, and $b-x:s::a:\frac{as}{b-x}=$ Sine of $\angle$ D; moreover, as $1:b::\frac{as}{x}:\frac{abs}{x}=$ BD, and $1:b::\frac{as}{b-x}:\frac{abs}{b-x}=$ AB: Hence by Euclid 47.1. we have $\overline{\frac{abs}{b-x}}\Big|^2+\overline{\frac{abs}{x}}\Big|^2=b^2$; or $\overline{x^2-bx}\big|^2-2a^2s^2\times\overline{x^2-bx}=a^2b^2s^2$; and $x^2-bx-a^2s^2=-as\sqrt{a^2s^2+b^2}$, and $x=\frac{1}{2}b-\sqrt{a^2s^2+\frac{1}{4}b^2-as\sqrt{a^2s^2+b^2}}=21,4285$; Hence $\frac{abs}{b-x}$ or AB $=30$, and $\frac{abs}{x}$, or BD $=40$, and consequently the Area $=600$.

PROBLEM XXXVIII.

A Ship sails in the North East Quarter until her Departure is 100 Miles, she still continues the same Course 30 Miles further, and then finds her whole Difference of Latitude made good to be 100 Miles: Quere her Course and Distance sailed.



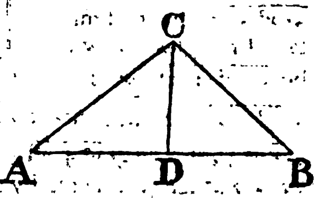
In the Right-Angled Triangle S E N, (in which S N denotes the Meridian of the Place sailed from) put SN = FG = 100 = a, GE = 30 = b; and SF = x, then will SG = $\sqrt{x^2+a^2}$, and by similar Triangles, it will be $\sqrt{x^2+a^2} : x :: \sqrt{x^2+a^2}+b : a$, hence $a\sqrt{x^2+a^2} = x\sqrt{x^2+a^2}+bx$, and $a\sqrt{x^2+a^2} - x\sqrt{x^2+a^2} = bx$; or $x^4-2ax^3+2a^2x^2-2a^3x+a^4=b^2x^2$; here, by adding a^2x^2 to both Sides, and extracting the Root, we have $x^2-ax+a^2=\sqrt{a^2+b^2}\times x$, put $2s=a+\sqrt{a^2+b^2}$, then $x^2-2sx=-a^2$; and $x=s-\sqrt{s^2-a^2}=81,103$, hence FN = $(100-81,103)=18,897$, and $18,897 : 30 :: 81,103 : 128,75$ = SG, therefore SE = 158,75 Miles.

OTHERWISE.

Put SG = x, then will SF = $\sqrt{x^2-a^2}$ and $x : \sqrt{x^2-a^2} :: x+b : a$, or $x\sqrt{x^2-a^2}+b\sqrt{x^2-a^2}=ax$, hence $x^4+2bx^3+\overline{b^2-a^2}\times x^2-2a^2bx-a^2b^2=a^2x^2$, or $\overline{x^2+bx}^2-2a^2\times\overline{x^2+bx}=a^2b^2$; and $x=\sqrt{a^2+\frac{1}{4}b^2+a\sqrt{a^2+b^2}}-\frac{1}{2}b$, consequently SE $=\sqrt{a^2+\frac{1}{4}b^2+a\sqrt{a^2+b^2}}+\frac{1}{2}b=158,75$ Miles, the whole Distance sailed, as before; and S G (128,75) : Rad. (1) :: FG (100) : 0,776699 the natural Sine of 50° 57′, the Angle FSG the Course, answering to N. E. ½ E.

PROBLEM XXXIX.

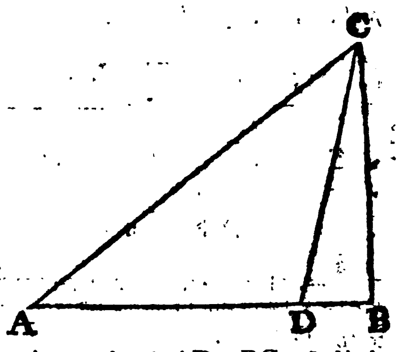
Having one Leg AC, of a Right-Angled Triangle ABC, to find the other Leg BC, so that the Hypothenuse AB shall be a mean Proportional between the Perpendicular CD falling thereon, and the Perimeter of the Triangle.



Put $AC = a$, and $BC = x$, then will $AB = \sqrt{x^2+a^2}$, and $CD = \frac{AC \times BC}{AB} = \frac{ax}{\sqrt{x^2+a^2}}$: therefore, by the Question, $x+a+\sqrt{x^2+a^2} : \sqrt{x^2+a^2} :: \sqrt{x^2+a^2} : \frac{ax}{\sqrt{x^2+a^2}}$; consequently $\frac{ax^2+a^2x}{\sqrt{xx+aa}} + ax = x^2+a^2$, or $\overline{ax^2+a^2x}^2 = \overline{x^2-ax+a^2}^2 \times \overline{x^2+a^2}$, hence $x^6 - 2ax^5 + 3a^2x^4 - 6a^3x^3 + 3a^4x^2 - 2a^5x + a^6 = 0$: Dividing this Equation by a^3x^3, we get $\frac{x^3}{a^3} - \frac{2x^2}{a^2} + \frac{3x}{a} - 6 + \frac{3a}{x} - \frac{2a^2}{x^2} + \frac{a^3}{x^3} = 0$, or $\overline{\frac{x}{a}+\frac{a}{x}}^3 - 2 \times \overline{\frac{x}{a}+\frac{a}{x}}^2 = 2$: Here putting $y+\frac{2}{3} = \frac{x}{a}+\frac{a}{x}$, we have $y^3 - \frac{4}{3}y - \frac{16}{27} = 2$, or $y^3 - \frac{4}{3}y = \frac{70}{27}$; this solved by the Theorem for Cubics, gives $y+\frac{2}{3} = \frac{1}{3}\sqrt[3]{35+\sqrt{1161}} + \frac{4}{3\sqrt[3]{35+\sqrt{1161}}} + \frac{2}{3} = 2{,}359304 = 2r = \frac{x}{a}+\frac{a}{x}$; hence $x^2+a^2 = 2arx$; and $x = a \times \overline{r \pm \sqrt{r^2-1}}$.

PROBLEM XL.

In the Plane Triangle ACD, there is given the Base AD $=57=a$, the Perpendicular CB falling from the Vertex on the Base produced $=55=b$; and the Angle ACD $=38^\circ$; to determine the Triangle.



Put the Sine to the Radius 1 of the Angle ACD $=s=,6156615$, its Co-sine $=c=,7880108$; and DB $=x$, then will AC $=\sqrt{x^2+2ax+a^2+b^2}$, and DC $=\sqrt{x^2+b^2}$: It is now well known that the Rectangle of the Sides AC and DC multiplied by the Sine of their included Angle, gives twice ($\frac{1}{2}$AD $\times$ BC $=\frac{1}{2}ab$) the Area of the Triangle ACD; hence $\sqrt{x^2+2ax+a^2+b^2} \times \sqrt{x^2+b^2} \times s=ab$; or $x^4+2ax^3+\overline{a^2+2b^2}\times x^2+2ab^2x+a^2b^2+b^4=\frac{a^2b^2}{ss}$, or $\overline{x^2+ax}|^2+2b^2\times\overline{x^2+ax}=\frac{a^2b^2}{ss}-a^2b^2-b^4$; and $x^2+ax+b^2=\frac{ab}{s}\sqrt{1-s^2}$:

But $\sqrt{1-s^2}$ is the Co-sine of the Angle ACD, therefore $x^2+ax+b^2=\frac{abc}{s}$; and $x=\sqrt{\frac{1}{4}a^2+\frac{abc}{s}-b^2}-\frac{1}{2}a=13,92484$.

Hence by Trigonometry, BC (55) : Rad. (1) :: BD (13,92484) : 0,2531789 the natural Tangent of $14^\circ\ 12'\ 27''$ the Angle BCD; and as (,2454343) the Sine of the Angle BCD : BD :: Rad 1. : DC $=56,735$: In the Right-Angled Triangle ABC, the Angle ACB ($=38^\circ+14^\circ\ 12'\ 27''$) $=52^\circ\ 12'\ 27''$; consequently the Angle A $=37^\circ\ 47'\ 33''$

33′, and as its Sine ,6128036 : BC 55 :: Rad. : AC = 89,751.

OTHERWISE.

Since $AB = \sqrt{\frac{1}{4}a^2 + \frac{abc}{s} - b^2} + \frac{1}{2}a = 70,92484$,

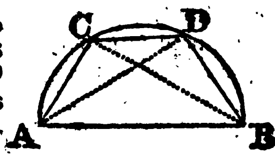
therefore $AC = \sqrt{\frac{1}{2}a^2 + \frac{abc}{s} + a\sqrt{\frac{1}{4}a^2 + \frac{abc}{s} - b^2}} =$

$89,7515$, and $\sqrt{x^2+b^2}$, or $DC =$

$$\sqrt{\frac{1}{2}a^2 + \frac{abc}{s} - a\sqrt{\frac{1}{4}a^2 + \frac{abc}{s} - b^2}} = 56,73536.$$

PROBLEM XLI.

Given three Sides of a Trapezium inscribed in a Semicircle, to find the Diagonals, and the remaining Side equal to the Diameter.



In the Semicircle ACDB, draw the Diagonals AD and BC; then will $AD \times BC = AB \times CD + AC \times BD$; and the Angles ACB and ADB are Right-Angles.

Put $AC = 15 = a$, $CD = 20 = b$, $BD = 25 = c$; $AB = x$, $AD = y$, and $BC = v$. Then $yv = bx + ac$, $y^2 = x^2 - c^2$, and $v^2 = x^2 - a^2$; by writing these Values of y^2 and v^2, in the Square of the first Equation, we have $\overline{x^2 - c^2} \times \overline{x^2 - a^2} = \overline{bx + ac}^2$, hence $x^3 - \overline{a^2 + b^2 + c^2} \times x = 2abc$; In Numbers $x^3 - 1250x = 15000$: Here by trying with Multiples of 10 (as has been directed) I readily find x equal to 40,3 nearly; then from the general Cubic Equation for converging Series, we have $a = 0$, $b = 1250$, $s = 15000$, and $r = 40,3$, and by taking only the first Power of z, we have $z = \frac{s - r^3 + br}{3r^2 - b} = -.021$, nearly, this added to 40,3, gives

$x=40,279$. Now put $r=40,279$; then n being $=3r^2-b$, and $t=s-r^3+br$, we have $\frac{t}{n}=,0000051797=p$; and $z=\frac{t-mp^2}{n}=.0000517976258$, and consequently $x=r+z=40,2790517976258=$ AB the Diameter.

Hence y, or AD $=\sqrt{x^2-c^2}=31,5816721$, and v, or BC $=37,3818407$.

Of unlimited Problems.

A QUESTION is said to be unlimited, when the Equations expressing the Conditions thereof, are fewer in Number than the unknown Quantities to be determined; such Problems will admit of innumerable Answers, if Fractions and negative Quantities be included: But the Answers in whole positive Numbers to which the Question is commonly restrained, are for the most Part limited to a determinate Number, as will appear by the following Examples.

LEMMA.

Suppose there were given the Equation $ay=bx\pm c$, to find x and y in whole positive Numbers.

Let w stand for the Words a whole Number: then dividing the given Equation by a, we have $y=\frac{bx\pm c}{a}=w$. per Question.

Now since (by the Question) x is a whole Number, it is evident that $\frac{ax}{a}$, $\frac{2ax}{a}$, $\frac{3ax}{a}$, &c. likewise $\frac{ax+a}{a}$, $\frac{ax+2a}{a}$, $\frac{2ax+a}{a}$, &c. are all whole Numbers; and either

either of these may be subtracted from any Multiple of the Value of y abridged; that is, from any Multiple of the fractional Part of $\frac{bx \pm c}{a}$ in its lowest Terms; or this from any of the foregoing Integers, or from any other whole Number, or that from this, &c. continually, till the Co-efficient of x is reduced to Unity, and you will have $\frac{x \pm n}{a} = wh. = p$, hence $x = ap \pm n$: Here p may be any whole Number taken at Pleasure.

This Operation is founded on these obvious Principles, that, whole Numbers subtracted from, or multiplied by whole Numbers produce whole Numbers; and whatever Number (a) measures the Whole and one Part of another, must evidently measure the remaining Part.

EXAMPLE I.

Given $14y = 19x + 11$, to find x and y Integers.

Here $y = \frac{19x + 11}{14} = x + \frac{5x + 11}{14} = wh.$ therefore $\frac{5x + 11}{14} = wh.$ this multiplied by 3, and $\frac{14x + 28}{14} = wh$ taken from the Product, leaves $\frac{x + 5}{14} = wh. = p$, hence $x = 14p - 5$. Here to have the least affirmative Value of x, take $p = 1$, then will $x = 14p - 5 = 14 - 5 = 9$, and from the given Equation we have $y = \frac{19x + 11}{14} = \frac{182}{14} = 13$.

And these two are the only positive Integers that can solve the Equation; because 19 the Co-efficient of x is greater than 13, the greatest Value of y; for having found the least Value of x, the greatest of y will be al-

 ways

ways obtained from the given Equation; or, having in like Manner determined the least Value of y, the greatest of x will become known from the Equation proposed; and the Number of Answers (when the Question admits of more than one) may be found by adding the Co-efficient of y continually to the least Value of x, and subtracting the Co-efficient of x from the greatest Value of y, so long as the Remainders will come out affirmative); and if the greatest Value of y be divided by the Co-efficient of x, the Quotient more 1 will be the Number of all the Answers, and the Remainder will be the least Value of y; but if the given Equation admits of a common Measure you must divide it thereby, so that the Co-efficients of x and y may become prime to each other.

EXAMPLE II.

Given $43x+34y=4000$, to find all the Values of x and y in positive whole Numbers.

Here $y=\frac{4000-43x}{34}=117-x+\frac{22-9x}{34}=wh.$

therefore $\frac{9x-22}{34}=wh.$ which, multiplied by 4, and $\frac{17x}{17}$ taken from the Product, leaves $\frac{36x-88}{34}-\frac{17x}{17}=\frac{18x-10}{17}-2-\frac{17x}{17}=\frac{x-10}{17}=wh.$ and consequently $\frac{x-10}{17}=wh.=p$, or $x=17p+10$, here by taking $p=0$, we have $x=10$, and $y=\frac{4000-43x}{34}=105$:

Now by continually adding 34, the Co-efficient of y, to 10 the least Value of x, and subtracting 43 the Co-efficient of x, from 105 the greatest Value of y, we have $\left.\begin{array}{l} x \\ y \end{array}\left\{\begin{array}{l} =10, 44, 78, \\ =105, 62, 19, \end{array}\right.\right\}$ Three Answers in whole Numbers.

Here

Here subtracting 43 from 105, there remains 62, and 43 taken from 62 leaves 19 the least positive Value of y; for if 43 be taken from 19 the next Remainder will be negative, which shews that the given Equation admits of but three Answers in whole positive Numbers; and the Values of x and y are in Arithmetical Progression.

EXAMPLE III.

Given $9x+13y=2000$, to find all the possible Values of x and y in whole positive Numbers.

Here $x=\frac{2000-13y}{9}=222-y+\frac{2-4y}{9}=wh.$

therefore $\frac{4y-2}{9}=wh,$ this multiplied by 2, and the Product taken from $\frac{9y}{9}$, leaves $\frac{y+4}{9}=wh.=p$, hence $y=9p-4$, take $p=1$, then will $y=9p-4=5$, and $x=\frac{2000-13y}{9}=215.$

Here by proceeding as in the last Example, you will have

$x=$	215.	202.	189.	176.	163.	150.	137.	124.	111.
$y=$	5.	14.	23.	32.	41.	50.	59.	68.	77.
	98.	85.	72.	59.	46.	33.	20.	7.	
	86.	95.	104.	113.	122.	131.	140.	149.	

Seventeen Answers in whole Numbers.

EXAMPLE IV.

Suppose $4y=14x+7$: Quere all the Values of x and y in positive Integers.

Here $y=\frac{14x+7}{4}=3x+1+\frac{2x+3}{4}=wh.$ per Question,

tion, therefore $\frac{2x+3}{4}=wh.$ from twice this take $\frac{4x}{4}$, and there will remain $\frac{6}{4}$, or $\frac{3}{2}$, which not being a whole Number, shews that x is no Integer; and to know when this will happen without trying the Operation, observe whether the Co-efficients of x and y, in the proposed Equation, admit of a common Divisor which will not measure the third Term; if they do, no Integers can be found for x and y from that Equation. Thus, in the present Case, 2 will measure the Co-efficients 4 and 14; but it will not measure the third Term 7.

EXAMPLE V.

It is required to pay 351 Pounds with Guineas and Moidores, so as to have the least Number of Pieces; and to find what the 351 Pounds will amount to, if paid every Way it will possibly admit of, which Sum is equal to a young Lady's Fortune; and which you are desired to find?

Put x for the Number of Guineas and y for the Moidores, then per Question $21x+27y=7020$ Shillings, or dividing by 3, we have $7x+9y=2340$; and $y=\frac{2340-7x}{9}$ $=260-\frac{7x}{9}=wh.$ therefore $\frac{7x}{9}=wh.$ from 4 Times this, take $\frac{27x}{9}$, there remains $\frac{x}{9}=wh.=p$, here taking $p=1$, we have $x=9$, and $y=260-\frac{7x}{9}=253$, hence $x+y\ (=9+253)=262$ the least Number of Pieces; because there are taken the most Moidores, except paying the Whole with them. And dividing the greatest Value of y, namely 253 by 7, the Co-efficient of x, the Quotient is 36, and 1 remains, therefore the least Value of y is 1, and the Number of Ansures if the Payment with one Moidore

Moidore be admitted is 37, in which Case $351\times37=12987$l. the Fortune required.

But if the Payment with one Moidore be not admitted, then there will be but 36 Ways to pay 351l. with Guineas and Moidores, in which Case the Fortune is 12636l.

LEMMA.

In two Equations, including three unknown Quantities, you must find the Limits of one of them by assuming each of the other two equal to Unity; and if there are four or more unknown Quantities, all their Limits, except two, must be found in the same Manner.

EXAMPLE VI.

A Vintner has Wines at 24d. 22d. and 18d. per Gallon; of which he would mix 30 Gallons, to be sold at 20d. the Gallon: How much must he take of each Sort?

Put x, y, and z respectively for the Quantities at 24d. 22d. and 18d. a Gallon, then per Question $x+y+z=30$, and $24x+22y+18z=30\times20=600$; from both 22 and 24 Times the first Equation, take the second, and you will have $-2x+4z=60$, and $2y+6z=120$, or $x=2z-30$, and $y=60-3z$: From these two Equations, assuming $x=y=1$, we get $z=\frac{31}{2}=15\frac{1}{2}$ and $z=\frac{59}{3}=19\frac{2}{3}$ the least and greatest Limits of z; therefore z is greater than $15\frac{1}{2}$ and less than $19\frac{2}{3}$; but z may be any Number between $15\frac{1}{2}$ and $19\frac{2}{3}$, and consequently 16, 17, 18, and 19, are all the possible Values of z in whole Numbers; which being successively written for z, in the Equations $x=2z-30$, and $y=60-3z$; you will have $x=2$, 4, 6, 8, and $y=12$, 9, 6, 3, the four Values of x and y: So that the Question admits of four Answers in whole Numbers, and no more, as appears from the Limits of z.

To find such a whole Number x, as being divided by given Numbers a, b, c, &c. shall leave given Remainders f, g, h, &c.

Subtract each of the given Remainders from x; and divide the Differences $x-f$, $x-g$, $x-h$, &c. by their respective given Divisors; then, if the Question proposed be possible, the several Quotients $\frac{x-f}{a}$, $\frac{x-g}{b}$, $\frac{x-h}{c}$ &c. must necessarily be all whole Numbers, because the Quantities f, g, h, &c. that would have remained by dividing x, the Number sought, by a, b, c, &c. are previously subtracted from it, before the Division is begun. Make the first of these whole Numbers equal to p, that is, assume $\frac{x-f}{a}=p$, then will $x=ap+f$, write this Value of x in the second whole Number, and reduce the Co-efficient of p to Unity, as you did that of x in the preceding Problems, then put this Quantity so reduced equal to q, from this Equation find the least Value of p and write it for p in the Equation $x=ap+f$, and write the Value of x thence arising in the third whole Number, in which reduce the Co-efficient of q to Unity, make this Number so reduced equal to r, and from this Equation find the least Value of q, which substitute for q in the foregoing Equation of x, and write this Value for x in the fourth whole Number and so on, as in the following Examples, which will illustrate this Rule.

EXAMPLE I.

Find the least whole Number, which being divided by 19 and by 28, shall leave 7 and 13 remaining.

Put x for the Number, then $\frac{x-7}{19}$ and $\frac{x-13}{28}$ are whole

whole Numbers; let $\frac{x-7}{19}=p$, then $x=19p+7$, this being written for x in $\frac{x-13}{28}$, gives $\frac{19p-6}{28}=wh.$ therefore $\frac{\overline{3\times 19p-6}}{28}=2p+\frac{p-18}{28}=wh.$ and $\frac{p-18}{28}=wh.=q$, hence $p=28q+18$, here, by taking $q=0$, we have $p=18$; and $x=19p+7=19\times 18+7=349$, the Number sought.

EXAMPLE II.

Find the least whole Number, which being divided by 8, 9, and 10, shall leave 6, 5, and 6, respectively remaining.

Put x for the Number, then $\frac{x-6}{8}$, $\frac{x-5}{9}$ and $\frac{x-6}{10}$ are whole Numbers, and $\frac{x-6}{8}=p$, or $x=8p+6$, hence $\frac{x-5}{9}$ (by writing $8p+6$ for x) becomes $\frac{8p+1}{9}$, and $\frac{9p}{9}-\frac{8p+1}{9}$ or $\frac{p-1}{9}=q$, therefore $p=9q+1$, this being written for p in the foregoing Equation $x=8p+6$, gives $x=72q+14$, writing $72q+14$ for x, in the third whole Number $\frac{x-6}{10}$, there arises $\frac{72q+8}{10}$, or $\frac{36q+4}{5}$ $=7q+\frac{q+4}{5}=wh.$ and $\frac{q+4}{5}=r$, or $q=5r-4$, here by taking $r=1$, we have $q=5r-4=1$, and $x=72q+14$ $=72+14=86$, the Number required.

OTHERWISE.

OTHERWISE.

Since two of the preceding Numbers, namely $\frac{x-6}{8}$ and $\frac{x-6}{10}$, have the same Numerator, this Numerator $x-6$ must therefore be divisible by 40, the least common Multiple of the Denominators 8 and 10: Hence then, we have $\frac{x-6}{40}=wh.=p$, and $x=40p+6$, therefore $\frac{x-5}{9}=\frac{40p+1}{9}=4p+\frac{4p+1}{9}=wh.$ and $\frac{9p}{9}-\frac{\overline{4p+1}\times 2}{9}$ or $\frac{p-2}{9}=q$, and $p=9q+2$, here, taking $q=0$, we have $p=2$, and $x=40p+6=40\times 2+6=86$, the very same as before.

EXAMPLE III.

Find the least whole Number, which being divided by 2, 3, 5, and 7, shall leave remaining 0, 2, 4, and 6.

Put x for the Number, then $\frac{x-0}{2}$, $\frac{x-2}{3}$, $\frac{x-4}{5}$ and $\frac{x-6}{7}$ are Integers, and $\frac{x-0}{2}=p$, or $x=2p$, writing $2p$ for x, in $\frac{x-2}{3}$ we have $\frac{2p-2}{3}$, and $\frac{3p}{3}-\frac{2p-2}{3}$, or $\frac{p+2}{3}=q$, or $p=3q-2$, hence $x=2p=6q-4$, writing $6q-4$ for x, in $\frac{x-4}{5}$, we have $\frac{6q-8}{5}=q-1+\frac{q-3}{5}wh.$ and $\frac{q-3}{5}=r$, or $q=5r+3$, hence x

$=6q$

$=6q-4=30r+14$, writing $30r+14$ for x, in the fourth whole Number $\frac{x-6}{7}$, it becomes $\frac{30r+8}{7}=4r+1+\frac{2r+1}{7}=wh.$ and $\frac{\overline{2r+1}\times 4}{7}-\frac{7r}{7}$, or $\frac{r+4}{7}=s$, and $r=7s-4$; here, by taking $s=1$, we have $r=7s-4=3$; and $x=30r+14=104$, the Number sought.

EXAMPLE IV.

Find the least whole Number, which being divided by 2, 3, 4, 5 and 6, shall always leave 1 remaining; but being divided by 7, 0 shall remain.

Put x for the Number, then $\frac{x-1}{2}$, $\frac{x-1}{3}$, $\frac{x-1}{4}$, $\frac{x-1}{5}$, $\frac{x-1}{6}$, and $\frac{x-0}{7}$ are to be all Integers.

Here five of the Numerators being equal, and the least common Multiple of the Denominators 2, 3, 4, 5, 6, being 60, we have $\frac{x-1}{60}=wh.=p$, or $x=60p+1$, by writing $60p+1$ for x, in $\frac{x-0}{7}$, we have $\frac{60p+1}{7}=8p+\frac{4p+1}{7}=wh.$ and $\frac{8p+2}{7}-\frac{7p}{7}$, or $\frac{p+2}{7}=q$, and $p=7q-2$, here taking $q=1$, we have $p=7q-2=5$, and $x=60p+1=301$, the Number required.

The

The investigation of Rational Squares, Cubes, &c.

PROBLEMS of this Sort frequently admit of an infinite Number of Answers; and yet none of the Quantities required can be taken at Pleasure, but must be investigated by putting one or more Letters for the Square, Cube, &c. which Letters, or Algebraic Squares, Cubes, &c. must be so assumed, that when an Equation containing two unknown Quantities is involved, either the given Number or the highest Power of one of the unknown Quantities may be on both Sides of the Equation, and destroyed out of it, so that you may have one of the unknown Quantities of but one Dimension, in this Equation, the other unknown Quantities may be taken at pleasure. But no general Rule can be given to suit all Cases, therefore the Resolution will often depend on the Art of the Analyst.

Note. A Square Number multiplied or divided by a square Number, produces a Square: And a Cube Number multiplied or divided by a Cube, produces a Cube Number, &c:

EXAMPLE I.

Find two Numbers in the Ratio of 5 to 4, whose Sum and Difference shall be both Square Numbers.

Put $5x$ for the greater and $4x$ for the less Number, then their Sum $9x$ and Difference x, being Squares by the Question, it is evident that $4x$ is a Square Number; assume $x^2=4x$, then will $x=4$, the Square of 2, and $9x=36$, the Square of 6; consequently $5x=20$ and $4x=16$, are two Numbers which will solve the Problem; for $20 : 16 :: 5 : 4$.

If you multiply x the Difference between $5x$ and $4x$ by any other Square Number; and put the Product equal to x^2, you will have two other Numbers which will Answer the Question.

EXAMPLE

EXAMPLE II.

Find two Square Numbers whose Difference is 40 (a).

Let x^2 and y^2 be the Numbers required.

Put $x^2 = \overline{\frac{z+v}{2}}\Big|^2$, and $y^2 = \overline{\frac{z-v}{2}}\Big|^2$; this Equation taken from the first, leaves $x^2 - y^2 = \frac{4vz}{4} = vz = a$, per Question; hence $z = \frac{a}{v}$: take $v = 4$, then $z = \frac{a}{v} = \frac{40}{4} = 10$: whence $x^2 = \overline{\frac{z+v}{2}}\Big|^2 = 49$, and $y^2 = \overline{\frac{z-v}{2}}\Big|^2 = 9$; consequently $x = \frac{z+v}{2} = 7$, and $y = \frac{z-v}{2} = 3$.

So that if the given Difference be resolved into any two unequal Factors, half the Sum and half the Difference of those Factors will be Sides of Squares having the given Difference: Thus $10 \times 4 = 20 \times 2 = 8 \times 5 = 40$, then $\frac{10+4}{2} = 7$; and $\frac{10-4}{2} = 3$; likewise $\frac{20+2}{2} = 11$, and $\frac{20-2}{2} = 9$; also $\frac{8+5}{2} = 6.5$, and $\frac{8-5}{2} = 1,5$, &c.

EXAMPLE III.

A Vintner mixed two sorts of Wine, some at 8 and some at 5 Pence per Pint: The Price in Pence of the whole Mixture is a Square Number, and if 60 be added thereto, the Sum will be a Square Number, whose Root is equal to the Number of Pints which he mixed of both Sorts: How much did he mix of each Sort?

Put x for the Number of Pints at 8 Pence per Pint; y for the Pints at 5 Pence each, and let a denote the given

U Number

Number 360; then $8x+5y$ will be the Price of the whole in Pence, which muſt be a Square, and by the Queſtion $\sqrt{8x+5y+a}$ equals $x+y$ (all the Pints he mixed) therefore $8x+5y+a$ is equal to $\overline{x+y}|^2$: Let $8x+5y=\overline{\frac{z-v}{2}}|^2$, and $8x+5y+a=\overline{x+y}|^2=\overline{\frac{z+v}{2}}|^2$: ſubtracting the firſt Equation from the ſecond, we have $a=vz$, and $z=\frac{a}{v}$; take $v=10$, then $z=\frac{a}{v}=36$: hence $\frac{36+10}{2}=23$ and $\frac{36-10}{2}=13$, are the Square Roots of the Quantities $\overline{x+y}|^2$ and $8x+5y$, whence $x+y=23$, and $8x+5y=\overline{13}|^2=169$; from theſe two Equations we get $x=18$, and $y=5$; hence $8x+5y+a=529$ the Square of 23; all which anſwers the Conditions of the Queſtion.

If you would find the Limits of v, write $\frac{a}{v}$ for z in the ſecond Equation, and you will have $x+y=\frac{\frac{a}{v}+v}{2}$ or $y=\frac{a+v^2}{2v}-x$, and by writing theſe Values of y and x in the firſt Equation there will ariſe $3x+\frac{5a+5v^2}{2v}=\frac{\overline{a-v^2}|^2}{4v^2}$; hence $x=\frac{\overline{a-v^2}|^2-10v\times\overline{a+v^2}}{12v^2}$, and $y\ (=\frac{a+v^2}{2v}-x)=\frac{a+v^2}{2v}\ \frac{\overline{a+v^2}|^2-10v\times\overline{a+v^2}}{12v^2}=\frac{16v\times\overline{a+v^2}-\overline{a-v^2}|^2}{12v^2}$.

Here it is obvious that x and y will be poſitive as long as the Numerators are affirmative; therefore v may be taken

ken equal to any Number which gives $\overline{a-v^2}|^2$ greater than $10v\times\overline{a+v^2}$ and less than $16v\times\overline{a+v^2}$.

LEMMA.

If a and b be the Legs of a Right-Angled Triangle, and h the Hypothenuse; then h^2+2ab and h^2-2ab are Square Numbers: For per Euclid's 1. 47. $h^2=a^2+b^2$.

Here, by adding $2ab$ to, and subtracting it from both Sides of the Equation, we have $h^2+2ab=a^2+2ab+b^2$, and $h^2-2ab=a^2-2ab+b^2$, two universal Squares.

EXAMPLE IV.

Find a Number, which being added to, and subtracted from its Square, both the Sum and Difference shall be Square Numbers.

Let a (96) denote the double Product of the Legs of a Right-Angled Triangle whose Hypothenuse is h (10); then since h^2+a and h^2-a, are Squares, multiply them by the Square x^2, and you will have $h^2x^2+ax^2$ and $h^2x^2-ax^2$ Squares; here, seeing ax^2 added to, and subtracted from the Square of hx, gives Squares, it is evident therefore, that ax^2 is equal to hx, the Number sought, hence $ax^2=hx$, or $x=\frac{h}{a}$, and consequently $ax^2=\frac{h^2}{a}=\frac{100}{96}=\frac{25}{24}$, a Number that will answer the Conditions of the Question: for $\overline{\frac{25}{24}}|^2+\frac{25}{24}=\frac{1225}{576}$ the Square of $\frac{35}{24}$, and $\overline{\frac{25}{24}}|^2-\frac{25}{24}=\frac{25}{576}$ the Square of $\frac{5}{24}$.

EXAMPLE V.

Find two such Numbers, that the Sum of their Cubes shall be a perfect Sursolid.

 Put

Put x and ax for the Numbers, then per Queſtion $x^5 = a^3 x^3 + x^3$, here dividing by x^3, we have $x^2 = a^3 + 1$: Let $a = 2$, and you will have $x^2 = 9$, or $x = 3$; hence the Numbers required are 3 and 6, the Sum of their Cubes is $3^3 + 6^3$, or $243 = 3^5$.

EXAMPLE VI.

Find two Numbers, ſuch that the Sum of their Squares, and the Sum of their Cubes ſhall be both Square Numbers.

Put x and y for the required Numbers, then $x^2 + y^2$ and $x^3 + y^3$ are to be Squares.

Let $y = \frac{3}{4}x$*, then will $x^2 + y^2 = x^2 + \frac{9}{16}x^2 = \frac{25x^2}{16}$ a Square, and $x^3 + y^3 = x^3 + \frac{27x^3}{64} = \frac{91x^3}{64}$ which muſt likewiſe be a Square; aſſume $\frac{91x^3}{64} = \overline{91}|^2 \times a^2 x^2$; this reduced, gives $x = 91 \times 64a^2$; here taking $a = \frac{1}{4}$, we have $x = 91 \times 4 = 364$, and $y = \frac{3}{4}x = 273$; and theſe two Numbers will anſwer the Queſtion.

For $\overline{364}|^2 + \overline{273}|^2 = \overline{455}|^2$,
and $\overline{364}|^3 + \overline{273}|^3 = \overline{8281}|^2$.

*Since $y = \frac{3}{4}x$, therefore $\frac{16y^2}{9} = x^2$, and $x^2 + y^2 = \frac{25x^2}{16} = \frac{25y^2}{9}$, conſequently $\sqrt{x^2 + y^2} = \frac{5}{4}x = \frac{5}{3}y$. See the Solution to Problem IX.

EXAMPLE VII.

Find two Numbers ſo, that their Difference, the Difference of their Squares, and the Difference of their Cubes, may be all Square Numbers.

Put

Put x for the greater and y for the less Number, then $x-y$, x^2-y^2 and x^3-y^3 must be all Squares, divide the second x^2-y^2 by the first $x-y$, so shall the Quotient $x+y$ be a Square also: Assume $x-y=a^2$, and $x+y=4a^2$, the Sum and Difference of these two Equations, gives $x=\frac{5}{2}a^2$, and $y=\frac{3}{2}a^2$: These Values of x and y being written in the third Square x^3-y^3, there arises $\frac{125a^6}{8}-\frac{27a^6}{8}$, or $\frac{49a^6}{4}$ a perfect Square, therefore the two Numbers sought are $\frac{5}{2}a^2$ and $\frac{3}{2}a^2$: By taking $a=2$, we have $x=\frac{5}{2}a^2=10$, and $y=\frac{3}{2}a^2=6$, by taking $a=4$, we have $x=\frac{5}{2}a^2=40$, and $y=\frac{3}{2}a^2=24$; when $a=6$, then $x=\frac{5}{2}a^2=90$, and $y=\frac{3}{2}a^2=54$, and thus by taking different Values for a, you may obtain as many Answers to the Question as you please, and all in Integers, when a is an even Number.

EXAMPLE VIII.

Find two Numbers in the Ratio of 7 to 3, so that each of them being added to the Square of their Sum, shall make a Square.

Put $a=7$, $c=3$, $a+c=10=s$; $ax=$ the greater, and $cx=$ the less Number, then the Square of their Sum will be $\overline{ax+cx}|^2=s^2x^2$, to which adding ax and cx separately, we have s^2x^2+ax, and s^2x^2+cx, which must be both Squares by the Question: assuming $s^2x^2+c=\overline{sx-v}|^2$, we have $x=\frac{v^2}{c+2sv}$, this being written for x in the foregoing Square s^2x^2+ax, it becomes $\frac{s^2v^4}{\overline{c+2sv}|^2}+\frac{av^2}{c+2sv}$, this multiplied by $\frac{\overline{c+2sv}|^2}{v^2}$, gives $s^2v^2+ac+2asv$, which must be a Square, let $s^2v^2+ac+2asv=\overline{sv+2c}|^2$, then will $ac+2asv=4csv+4c^2$, and $v=\frac{4c^2-ac}{2as-4cs}=,75$, hence $x=$

$\frac{v^2}{c+2sv}=,03125$, therefore $ax=\frac{av^2}{c+2sv}=,21875$, and $cx=\frac{cv^2}{c+2sv}=,09375$, and these two Numbers will solve the Problem.

EXAMPLE IX.

Find three such Numbers that the Sum and the Difference of every two of them shall each be a Square Number.

Assume $x^2z^2+y^2v^2$, $2xyzv$ and $x^2v^2+y^2z^2$, for the three required Numbers, these evidently answer four Conditions of the Question; but $x^2z^2+y^2v^2+x^2v^2+y^2z^2$, or $\overline{x^2+y^2}\times\overline{z^2+v^2}$ the Sum, and $x^2z^2+y^2v^2-x^2v^2-y^2z^2$, or $\overline{x^2-y^2}\times\overline{z^2-v^2}$ the Difference between the first and third Numbers must be also Squares: In these, by writing $r+s$, $r-s$, $p+q$ and $p-q$ respectively for x, y, z, and v, we have $\overline{2r^2+2s^2}\times\overline{2p^2+2q^2}$, and $4rs\times4pq$, or $\overline{r^2+s^2}\times\overline{p^2+q^2}$, and $rspq$, which must be Squares; assume $rs=pq$, or $r=\frac{pq}{s}$, and $c^2=p^2+q^2$, then by writing $\frac{pq}{s}$ for r, and c^2 for p^2+q^2, in $\overline{r^2+s^2}\times\overline{p^2+q^2}$, we have $\overline{\frac{p^2q^2}{s^2}+s^2}\times c^2$, or $p^2q^2+s^4$, which must be a Square; Put $p^2q^2+s^4=\overline{\tfrac{1}{2}c^2-s^2}|^2$, then will $s^2=\frac{c^4-4p^2q^2}{4c^2}$; Here by writing $\overline{p^2+q^2}|^2$ for c^4 in the Numerator, we have $s^2=\frac{p^4-2p^2q^2+q^4}{4c^2}$; or $s=\frac{p^2-q^2}{2c}$; Here p must be greater than q, and p and q must be taken so as c, or its equal $\sqrt{p^2+q^2}|$ becomes rational; and this will be the Case when $\frac{5}{4}p$ is equal to $\frac{5}{3}q$: For then $c\,(=\sqrt{p^2+q^2}|)=\frac{5}{4}p=\frac{5}{3}q$, by Problem VI. Take $p=4$, and $q=3$, then $c=\frac{5}{3}q=$,

$\frac{5}{3}q=5$, hence $s=\frac{p^2-q^2}{2c}=\frac{7}{10}$, and $r=\frac{pq}{s}=\frac{120}{7}$; whence $x=r+s=\frac{1249}{70}$, $y=r-s=\frac{1151}{70}$, $z=p+q=7$, and $v=p-q=1$. Now by ſubſtituting theſe Values of x, y, z, and v, in the original aſſumed Numbers, there will ariſe $\frac{77764850}{4900}$, $\frac{20126386}{4900}$, and $\frac{66475250}{4900}$ three Numbers anſwering the Conditions of the Queſtion: But the common Denominator being a Square Number it may therefore be rejected, and ſo you will have 77764850, 20126386 and 66475250, three whole Numbers, which will ſolve the Problem.

EXAMPLE X.

Find a Number, to which adding a given Cube Number, the Sum ſhall be a Cube; and ſubtracting another given Cube Number the Remainder ſhall be a Cube.

Put x for the Number ſought, a^3 (8) and c^3 (1) for the given Cubes; aſſume $x+a^3=\overline{a+\frac{c^2v}{a^2}}\Big|^3=a^3+3c^2v+\frac{3c^4v^2}{a^3}+\frac{c^6v^3}{a^6}$, and $x-c^3=\overline{v-c}\big|^3=v^3-3cv^2+3c^2v-c^3$: then ſubtracting the ſecond Equation from the firſt, we have $v^3-\frac{c^6v^3}{a^6}=3cv^2+\frac{3c^4v^2}{a^3}$, or $a^6v^3-c^6v^3=3a^6cv^2+3a^3c^4v^2$, this Equation divided by $a^3v^2+c^3v^2$, gives $a^3v-c^3v=3a^3c$, and $v=\frac{3a^3c}{a^3-c^3}=\frac{24}{7}$: But $x-c^3=\overline{v-c}\big|^3$, hence $x=\overline{v-c}\big|^3+c^3=\overline{\frac{3a^3c}{a^3-c^3}-c}\Big|^3+c^3=\frac{5256}{343}$, a Number which will ſolve the Queſtion.

 It

It is evident in this Conclusion, that if c had not been given, it might have been taken equal to any Number less than a.

Problems in Experimental Philosophy.

PROBLEM I.

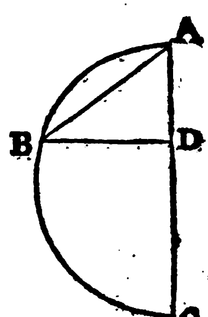
BEING on the Top of a Tower, I let fall a heavy Body to the Ground, and observed the Time elapsed between its leaving my Hand and the Return of the Sound, to be just four Seconds: Required the Height of the Tower.

Sound moves 1142 Feet per Second, heavy Bodies fall $16\frac{1}{12}$ Feet in the first Second, and the Space described by a falling Body is as the Square of the Time from the beginning of its Fall: Put $a=16\frac{1}{12}$, $s=1142$, $c=4$ Seconds; and $x=$ the Tower's Height, then $a : 1^{2''} :: x : \frac{x}{a} =$ the Square of the Time in which the Descent was made, therefore $\sqrt{\frac{x}{a}}$ is the Time of the Body's Descent; and $s : 1'' :: x : \frac{x}{s} =$ the Time of the Sound's Ascent, but the Time of the Body's descent, together with that of the Sound's Ascent, is equal to the whole Time elapsed, hence per Question $\frac{x}{s}+\sqrt{\frac{x}{a}}=c$, or $x+\frac{s}{2\sqrt{a}}\times x^{\frac{1}{2}}=cs$, here, completing the Square, &c. we have $\sqrt{x}+\frac{s}{2\sqrt{a}}=\sqrt{\frac{s^2}{4a}+cs}$, and consequently $x=\left(\sqrt{\frac{s^2}{4a}+cs}-\frac{s}{2\sqrt{a}}\right)^2=231.87$ Feet.

PROBLEM

PROBLEM II.

Suppose two Balls, each 10lbs. Weight were let descend at the same Moment of Time; one freely from the Top of a Tower, and the other on an inclined Plane; they arrive also at their respective Bottoms at the same Moment of Time when the Expression of the Sum of their Momentums was $3096\frac{2}{3}$ lbs. and the Difference between the Tower's Height and Plane's Length is 229 Feet. Required the Height of the Tower, Length of the Plane, Time of Descent, and the Momentum of each Ball?



Momentum signifies Force, and, in this Case, is equal to the Weight of a Ball, multiplied into its Velocity—Velocity denotes Swiftness, and is equal to the Quotient of the Space divided by the Time: and since the Balls arrived at their respective Bottoms at the same Instant, therefore let the Tower AC and the Plane AB be represented by the Diameter of Circle and a Chord thereof respectively; for a heavy Body will fall through any Chord of a Circle in the same Time that it would descend Perpendicularly through its Diameter. Make BD perpendicular to AC, put $a=16\frac{1}{12}$, $x=$ the Time of Descent, and $y=$ the Length of the Plane; then will $ax^2=$AC the Tower's Height, and ax is the Velocity acquired in x Seconds, but the Velocity acquired at the End of the Fall, is such as would carry a Body uniformly through twice the Space AC in the Time x, therefore $10\times 2ax$, or $20ax$ is the Momentum of the Ball which fell from the Tower: But

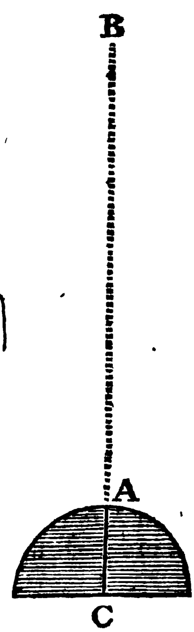
AC : AB :: AB : AD; that is, $ax^2 : y :: y : \frac{y^2}{ax^2}$ therefore $\overline{\frac{y^2}{a^2x^2}}\Big|^{\frac{1}{2}}$, or $\frac{y}{ax}$ is the Velocity acquired by falling from A to D, and is equal to that gained by falling from A to B, but the Velocity at the End of the Fall will be

$\frac{2y}{ax}$, therefore the Momentum of the Ball descending down the Plane will be $\frac{20y}{ax}$: hence per Question, $20ax + \frac{20y}{x} = 3096\frac{2}{3} = b$, and $ax^2 - y = 229 = d$: From these two Equations we get $x = \frac{b}{80a} + \sqrt{\overline{\frac{b}{80a}}\Big|^2 + \frac{d}{2a}} = 6$ Seconds; and $y\ (= ax^2 - d) = 36a - d = 350$ Feet the Length of the Plane; Hence ax^2 the Tower's Height is 579 Feet, and the Momentum of the Balls 1930 and $1166\frac{2}{3}$ lb. respectively.

PROBLEM III.

Suppose (as in the 67th Arithmetical Question) that the Earth's Radius is 4000 Miles, and that a Body on its Surface weighs 972 Pounds; to what Height must it be carried in the Air to weigh but 27 Pounds?

In the annexed Figure, let AC denote the Earth's Radius, and AB the Height required: Put $a = 972$, $c = 27$, $r = 4000 = AC$; and $x = AB$, then will $BC = x + r$, and since the Weights of Bodies above the Earth's Surface continually diminish, as the Squares of their Distances from its Centre increase, it will be as $c : a :: AC^2 : BC^2$, that is, as $c : a :: r^2 : \overline{x+r}|^2$, hence $\overline{x+r}|^2 \times c = ar^2$; and $x = r\sqrt{\frac{a}{c}} - r = 20000$ Miles, the Height sought.



PROBLEM IV.

Find the Length of a Pendulum that ſhall vibrate as many Times in a Minute, as it is Inches long.

A Pendulum that ſwings Seconds is 39,2 Inches long, and the Lengths of Pendulums are in the reciprocal Duplicate Proportion of the Number of Vibrations made in the ſame Time; therefore, putting x for the Length in Inches of the required Pendulum, it will be as $60 \times 60 : 39{,}2 :: x^2 : x$, hence $x = \frac{60 \times 60 \times 39{,}2}{x^2}$, and $x^3 = 141120$, therefore $x = \sqrt[3]{141120} = 52{,}063$ Inches.

PROBLEM V.

Suppoſe the mean Diſtance of the Earth from the Sun to be 95000000 of Miles, and that the Earth and the Planet Jupiter revolve round the Sun in 365,25 and 4332,5 Days reſpectively: Required the mean Diſtance of Jupiter from the Sun?

Having the Times of the periodical Revolutions of the Planets, and the mean Diſtance of any one of them from the Sun, the mean Diſtances of all the reſt (in our Syſtem) may be determined according to Kepler, and Newton, by this Proportion, namely, as the Square of the Time in which any one Planet makes a Revolution round the Sun, is to the Cube of its Diſtance from the Sun; ſo is the Square of the Time in which any other Planet is in revolving round the Sun, to the Cube of its Diſtance from the Sun: Hence then, by putting $a = 365{,}25$, $b = 4332{,}5$, $c = 95000000$; and $x =$ the Diſtance ſought, we have $a^2 : c^3 :: b^2 : x^3$, or $a^2 x^3 = b^2 c^3$, and $x = c\sqrt[3]{\frac{b^2}{a^2}} =$ 494108702 Miles, the required Diſtance of Jupiter from the Sun.

Problems producing Exponential Equations.

PROBLEM I.

IN what Time will the Amount of an Annuity of 100l. a Year become equal to the Amount of 1200l. put out at Interest; allowing 5 per Cent. per Annum compound Interest in both Cases?

Put $u=100$, $p=1200$, $r=1,05$, and $t=$ the Time required, then $\frac{ur^t-u}{r-1}$ is the Amount of 100l. for the Time t, and pr^t is the Amount of 1200l. put out at Interest for the same Time; hence, per Question, $\frac{ur^t-u}{r-1}=pr^t$, and $ur^t-u=rpr^t-pr^t$, or $ur^t+pr^t-rpr^t=u$; Put $q=u+p-rp$ $(=40)$, then will $qr^t=u$, or $r^t=\frac{u}{q}=2,5$, this Equation in Logarithms, becomes $t\times\text{Log. }r=\text{Log. }2,5$, hence $t=\frac{\text{Log. }2,5}{\text{Log. }r}=\frac{,3979400}{0,0211893}=18,7802334$ Years, the Time sought.

PROBLEM II.

In what Time will the Compound Interest of any Sum of Money put out at 5 per Cent. per Annum, become equal to the Principal?

Put $r=1,05$, $P=$ the Principal; and $n=$ the Number of Years sought, then $P\times r^n$ is the Amount of P in the Time n, and $P\times r^n-P$ is the Interest of P for the same Time n, hence per Question, we have $P\times r^n-P=P$, or $P\times r^n=2P$, or $r^n=2$; and $n=\frac{\text{Log. }2}{\text{Log. }r}=\frac{0,3010300}{0,0211893}=14,20669$ Years, the Time required.

PROBLEM

PROBLEM III.

If any propofed Sum of Money be put to Compound Intereft at 5 per Cent. per Annum, in what Time will the laft Year's Intereft become equal to the Principal?

Let r, P and n reprefent the fame Things as in the laft Problem; then Pr^n is the Amount of the Principal P in n Years, and Pr^{n-1} is the Amount of P for the whole Time (n) except the laft Year, and confequently $Pr^n - Pr^{n-1}$ is the laft Year's Intereft, hence per Queftion, we have $Pr^n - Pr^{n-1} = P$, and $r^n - r^{n-1} = 1$, or $r^{n-1} \times \overline{r-1} = 1$, therefore $r^{n-1} = \frac{1}{r-1}$, or $r^n = \frac{r}{r-1} = 21$; hence n, $= \frac{\text{Log. } 21}{\text{Log. } r} = \frac{1,3222193}{0,0211893} = 62,400329$ Years, the Time fought.

PROBLEM IV.

A has an Annuity of 100l. a Year, B puts out 1200l. at Intereft; in what Time will the Amount of A's Annuity exceed the Amount of B's 1200l. by 1457,554l. at the Rate of 5 per Cent. per Annum Compound Intereft?

Put $u = 100$, $p = 1200$, $d = 1457,554$, $r = 1,05$, and $n =$ the Time required, then by the Queftion, we have $\frac{ur^n - u}{r-1} - pr^n = d$, or $ur^n - u - rpr^n + pr^n = rd - d$, and $r^n = \frac{rd + u - d}{p + u - rp} = 4,3219425$, therefore $n = \frac{\text{Log. } 4,3219425}{\text{Log. } r} = \frac{0,6356790}{0,0211893} = 30$ Years.

PROBLEM V.

A agreed to lend B 720l. for a certain Time at 5 per Cent. per Annum Compound Intereft: But when juft half

half the Time was expired, B pays A the Principal and Intereſt together, and finds the Amount leſs than it would have been if he had kept the Money the whole Time agreed for by 225l. How long had B the Money in Hand?

Put $a=720$, $b=225$, $r=1{,}05$, and $x=$ the Time ſought, then ar^x will be equal to what A received in all, and ar^{2x} is equal to what A would have received if B had kept the Money the whole Time agreed for, hence per Queſtion we have $ar^{2x}-ar^x=b$, or $r^{2x}-r^x=\frac{b}{a}$; and by completing the Square, &c. we have $r^x-\frac{1}{2}=\sqrt{\frac{b}{a}+\frac{1}{4}}$, therefore $r^x=\sqrt{\frac{b}{a}+\frac{1}{4}}+\frac{1}{2}=1{,}25$, and conſequently $x=\frac{\text{Log. } 1{,}25}{\text{Log. } r}=4{,}5735$ Years.

PROBLEM VI.

If 200l. be due 3 Years hence, and 80l. five Years hence, in what Time muſt both be paid together at 5 per Cent. Compound Intereſt?

Here $\frac{200}{\overline{1.05}|^3}+\frac{80}{\overline{1.05}|^5}=234{,}4496$ the Sum of the preſent Worths, therefore $\frac{280}{234.4496}=1.189214$, the Amount of 1 Pound for the Time required, for which Time put x, and you will have $\overline{1.05}|^x=1.189214$, and $x=\frac{\text{Log. } 1.189214}{\text{Log. } 1.05}=3.55178$ Years.

PROBLEM VII.

A Company of Sailors dividing a Prize equally among them, found the Share of each in Pounds equal to twice the

the Number of Men; and that if one Pound were counted to the first Man, two to the second, four to the third, &c. till the Terms become equal in Number to the Men, the last Term would exactly contain all the Pounds in the Prize: Quere the Prize and Number of Men?

Put x for the Number of Men, and y for the Prize in Pounds, then per Question $2x = \frac{y}{x}$, or $2x^2 = y$: and the last Term of the Geometrical Series 1, 2, 4, 8, &c. continued to x Terms is $1 \times 2^{x-1}$, or 2^{x-1} equal to the Prize per Question, therefore $2x^2 = 2^{x-1}$, or $4x^2 = 2^x$: Here x must be an Integer by the Nature of the Question, and since $4x^2$ is a Square Number, therefore its equal 2^x, is a Square Number, but no odd Power of 2 is a Square Number, consequently x is an even Number, comprized in this Progression 2, 4, 6, 8, 10, &c. from which I find $x = 8$; hence y, or $2^{x-1} = 2^7 = 128$l.

PROBLEM VIII.

In what Time will any Sum at 5 per Cent. Simple Interest, amount to the same as it would do, if put out for the same Time at 4 per Cent. Compound Interest?

Put $a = .05$, $r = 1.04$, $p =$ the Principal; and $x =$ the Time required, then the Amount at Simple Interest is $apx + p$, and at Compound Interest it is pr^x, these are equal by the Question, hence $pr^x = apx + p$, therefore $r^x - ax = 1$, or $\overline{1.04}|^x - .05x = 1$: Here by Trial I find x very near 12, and writing 11.9 for x, we have $\overline{1.04}|^x - .05x = \overline{1.04}|^{11.9} - .595 = .999767$, which should be equal to 1, therefore the Error is .000233 in Defect; and writing 11.91 for x, the Error is .000111 in Defect, whence .000122 : 0.01 :: 0,000111 : 0.009, this added to 11.91 gives 11.919 which being written for x the Error is .000002 in Excess, then .000113 : 0.009 :: 0.000002 : 0.00015, this taken from 11.919, gives $x = 11.91885$ equal to 11 Years and 335 Days.

PROBLEM

PROBLEM IX.

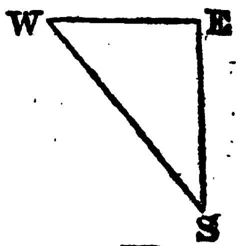
Find the Time in which x Pounds being put out at x per Cent. Compound Interest for x Years, will gain x in the Time x.

Here x being $=p=r=$ the Principal, Rate, and Time, it will be $100 : 100+x :: 1 : 1+\frac{x}{100}$, the Amount of 1 Pound for a Year, consequently $x\times\overline{1+\frac{x}{100}}|^{x}$ is the whole Amount of x Pounds for x Years; therefore $x\times\overline{1+\frac{x}{100}}|^{x}$ $-x$ is the Interest, which must, by the Question be equal to the Principal x, hence $x\times\overline{1+\frac{x}{100}}|^{x}-x=x$, or $x\times\overline{1+\frac{x}{100}}|^{x}=2x$; therefore $\overline{1+\frac{x}{100}}|^{x}=2$: Here I find x equal to 8.5, nearly, and writing 8.5 for x we have $\overline{1+\frac{x}{100}}|^{x}=\overline{1.085}|^{8.5}=2.000564$ which should have been equal to 2, therefore the Error is .000564 in Excess; and writing 8.499 for x the Error is .000244 in Excess, hence .00032 : 0.001 :: 0.000244 : 0.00076, this taken from 8.499, gives $x=8.49824$ Years.

PROBLEM X.

Two Ships, A and B sailed both at the same Time from one Place in 27° South Latitude: A steered due North 6 Miles the first Day, 10 the Second, 14 the third, 18 the fourth, &c. B steering between the North and West, sailed the first Day 8 Miles. the second 12, the third 18, the fourth 27, &c, till they both arrived in one Latitude, and were by Plane Sailing 631.3 Miles asunder: Required their

their Latitude arrived in, likewise B's Course and Distance sailed.



In the Right-Angled Triangle E W S, make E W $=631.3$ Miles $=$ B's Departure, then will S denote the Place from which the Ships sailed; put x for the Number of Terms or Days in which they sailed, then by Arithmetical Progression we have $2x^2+4x=$ S E, their Difference of Latitude $=$ the Distance A sailed, and by Geometrical Progression $16\times\overline{1.5}|^x-16$ $=$ S W $=$ the Distance B sailed, hence per Euc. 47.1, we have $\overline{16\times\overline{1.5}|^x-16}|^2=\overline{2x^2+4x}|^2+\overline{631.3}|^2$, or $\overline{1.5}|^x-\frac{1}{16}\sqrt{\overline{2x^2+4x}|^2+\overline{631.3}|^2}=1$: Here by Trial, I find x something less than 9.3, and writing 9.25 for x, we have $\overline{1.5}|^x-\frac{1}{16}\sqrt{\overline{2x^2+4x}|^2+\overline{631.3}|^2}=\overline{1.5}|^{9.25}-\frac{1}{16}\sqrt{441855,705625}$ $=42.5446-41.54514=.99946$, which should be Unity, so that the Error is .00054 in Defect; but writing 9.2502 for x the Error is .0027 in Excess, and .00324 : 0.0002 :: 0.00054 : 0.000033, this added to 9.25 gives $x=$ 9.250033; hence S E $=$ 208.126353 Miles, therefore the required Latitude is 23° 41′ 53″ South. B's Distance S W $=$ 664.72270069 Miles; and 664.72270069 : Radius :: 631.3 : 71° 45′ 13″ the Sine of B's Course, answering to W. N. W. $\frac{1}{3}$ W. nearly.

OTHERWISE.

From the Equation $\overline{16\times\overline{1.5}|^x-16}|^2=\overline{2x^2+4x}|^2+\overline{631.3}|^2$, we get $\overline{\overline{1.5}|^x-1}|^2-\dfrac{\overline{2x^2+4x}|^2+\overline{631.3}|^2}{256}=0$;

In this Equation by writing 9.25 for x, we have $\overline{41.5446}|^2$ $-441855.705625\div256=-0.04505$ the Error in defect, and by repeating the Operation the Value of x will be found $=9.250033$, as above.

X PROBLEM

PROBLEM XI.

Quere, the Value of x, when $4x^{3x}+x^x$ denotes the Length, and $4x^{3x}+x^x-260$ the Breadth of a Right-Angled Parallelogram whose Area is $9x^{6x}-16900$?

By the Rule for measuring a Parallelogram, we have $\overline{4x^{3x}+x^x}|^2-260\times\overline{4x^{3x}+x^x}=9x^{6x}-16900$ the Area per Question, whence by completing the Square, &c. we have $4x^{3x}+x^x-130=3x^{3x}$, or $x^{3x}+x^x-130=0$: Here x^x may be found by the general Rule for Cubics on Page 193, but this Equation is divisible by x^x-5, without Remainder, consequently $x^x=5$, and $x-\overline{5}|^{\frac{1}{x}}=0$: Here I find x equal to 2.13 nearly, and by writing 2.13 for x, we have $2.13-\overline{5}|^{\frac{1}{2.13}}=.001102$ the Error in Excess; but by writing 2.129 for x, the Error is .000653 in Defect, and .001755 : 0.001 :: 0.000653 : 0.000372, this added to 2.129, gives $x=2.129372$.

Astronomical Problems.

NOTE.

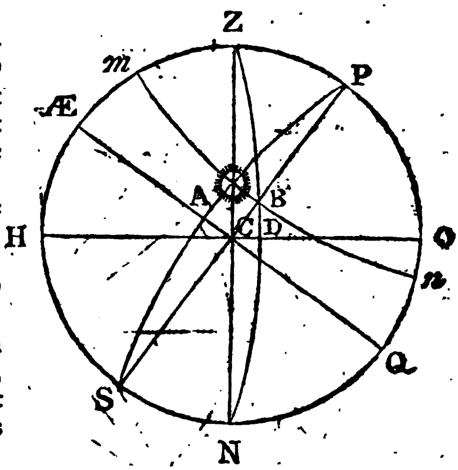
In solving these Problems, I shall use the Natural Sines, Tangents, &c. to the Radius 1.

EXAMPLE I.

IN a certain Northern Latitude the Sun's Azimuth, at 6 o'Clock in the Morning, was found to be 13° 20′ from the East Northward; it was also observed that the Sun was due East 6 Minutes after 7 o'Clock the same Morning: Quere, the Latitude of the Place, and the Sun's Declination?

In

In the Triangles AC☉, BCD Right-angled at A and D, put $s=,2840153$ the Sine of AC = 16° 30′ the Time from 6; $t=,2370044$ the Tangent of CD = 13° 20′ the Sun's Azimuth at six o'Clock; $x=$ the Tangent of A☉ the Sun's Declination,



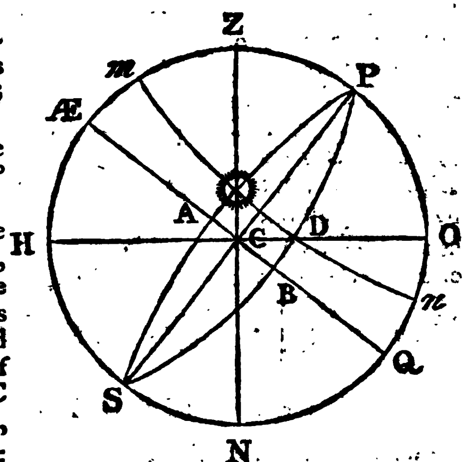
then $\frac{1}{x}$ is the Co-Tangent of A☉ or of BC; let y = the Sine of the < AC☉ the Latitude, then $\sqrt{1-y^2}$ and $\frac{\sqrt{1-y^2}}{y}$ will be the Co-Sine and Co-Tangent of the Latitude, and in the Triangles AC☉ and BCD, it will (by Spherical Trigonometry) be as $x : 1 :: s : \frac{\sqrt{1-y^2}}{y}$ hence $x\sqrt{1-y^2}=sy$, and as $\frac{1}{x} : 1 :: \sqrt{1-y^2} : t$, or $x\sqrt{1-y^2}=t$, therefore $sy=t$, or $y=\frac{t}{s}=,8344775$, the Sine of 56° 33′ 41″ the Latitude required: Now 90° − 56° 33′ 41″ = 33° 26′ 19″, and, as the Tangent of 33° 26′ 19″ : Rad. :: Sine of 16° 30′ : the Tangent of 23° 16′ = A☉ the Sun's Declination.

EXAMPLE II.

On a certain Morning the Sun's Amplitude was observed to be just 30° from the East towards the North; and

his

his Altitude on the Prime Vertical the same Morning was found to be 23° 30′: Quere, the Latitude of the Place, aud the Sun's Declination?



In the Right-angled Triangles A C D, and B CD; let $r=$,3960798 the Sine of C ☉ = 23° 20′ the Sun's Altitude on the Prime Vertical, $s=,5$ the Sine of CD = 30° his Amplitude; and $x=$ the Sine of the Angle A C ☉ the Latitude, then $\sqrt{1-x^2}=$ Sine of the < BCD, the Co-Latitude, and by Spherics it will be as $1 : r :: x : rx=$ A ☉ the Sun's Declination, and as $1 : s :: \sqrt{1-x^2} : s\sqrt{1-x^2}=$ BD the Sun's Declination also, therefore $rx=s\sqrt{1-x^2}$ or $r^2x^2=s^2-s^2x^2$, and $x=$

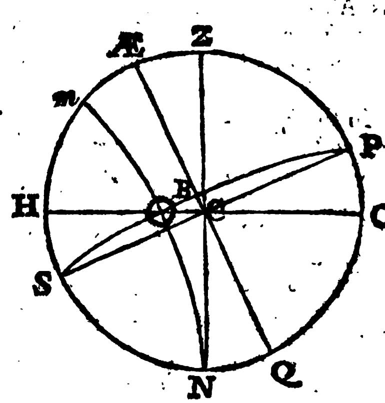
$\frac{s}{\sqrt{r^2+s^2}}=,7838578$ the Sine of 51° 37′ the Latitude sought, and rx, or A ☉ $=\frac{rs}{\sqrt{r^2+s^2}}=,3104792$ the Sine of 18° 5′ the Sun's Declination.

EXAMPLE III.

What Latitude is that, which is equal to the Sun's Declination on a certain Day when his Amplitude is 22° 34′ 24″ from the East Southward?

In

In the Triangle BC ☉, put $a = ,3838656$ the Sine of C ☉ the Sun's Amplitude; and $x =$ the Sine of the Latitude required, then $\sqrt{1-x^2} =$ the Sine of the $<$ BC ☉; and, as $1 : a :: \sqrt{1-x^2} : x$ (B ☉), by the Question, hence $a\sqrt{1-x^2} = x$, or $a^2 - a^2x^2 = x^2$, and

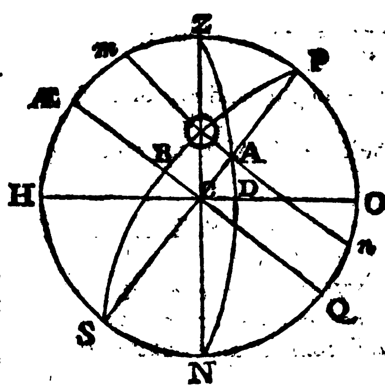


$x = \sqrt{\dfrac{a^2}{a^2+1}} = ,358369$ the Sine of 21°.

EXAMPLE IV.

'One Morning in Summer, before the Solstice, the Sun's 'Azimuth at 6 o'Clock was observed to be 9° 55′ from the 'East Northward, and his Altitude when due East was 19° '56′: Quere the Latitude of the Place, and the Sun's Declination?

In the Triangles BC, ☉ ACD, put $s = ,3409265$ the Sine of C ☉ the Sun's Altitude when East, $t = ,1748277$ the Tangent of C D, the Sun's Azimuth at 6; and $x =$ the Sine of the Latitude, then in the Triangle B C ☉ it will be as $1 : s :: x : sx =$ the Sine of B ☉ the Sun's Declination,



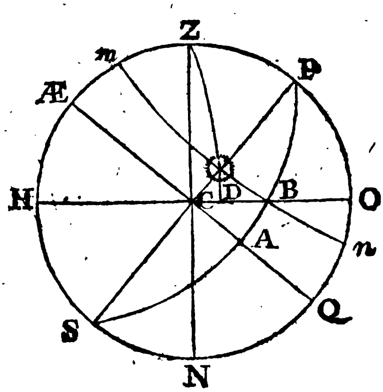
 therefore

therefore $\frac{\sqrt{1-s^2x^2}}{sx}$ is the Co-Tangent of B ☉, or of its equal AC, hence then in the Triangle ACD, we have $t:\sqrt{1-x^2}::1:\frac{\sqrt{1-s^2x^2}}{sx}$, therefore $\sqrt{1-x^2}=\frac{t\sqrt{1-s^2x^2}}{sx}$, hence $s^2x^2-s^2x^4=t^2-s^2t^2x^2$, or $x^4-\overline{t^2+1}\times x^2=-\frac{t^2}{s^2}$:

Put $2b=t^2+1$, then $x^4-2bx^2=-\frac{t^2}{s^2}$; hence $x=\sqrt{b+\sqrt{b^2-\frac{t^2}{s^2}}}=,7521853$ the Sine of 48° 56′ 48″ the Latitude required, and sx, or B☉ $=s\sqrt{b+\sqrt{b^2-\frac{t^2}{s^2}}}=,2564399$ the Sine of 14° 51′ 32″ the Sun's Declination.

EXAMPLE V.

In a certain Place in North Latitude, the Sun was observed to rise exactly at four o'Clock, and at six o'Clock his Altitude was taken the same Morning, and found to be 17° 30′: Required the Latitude of the Place, and the Sun's Declination.



In the Right-Angled Triangles ABC, CD☉; put $a=,3007058$ the Sine of D☉ the Sun's Altitude at 6, $d=,5$ the Sine of AC his Ascensional Difference; and $x=$ the Sine of the Latitude, then $\frac{x}{\sqrt{1-x^2}}$ is the Tangent of the (<☉CD) the Latitude; But as

$x : a :: 1 : \frac{a}{x}$ the Sine of C ☉, or of A B the Sun's Declination, therefore $\frac{\frac{a}{x}}{\sqrt{1-\frac{a^2}{x^2}}}$, or $\frac{a}{\sqrt{x^2-a^2}}$ is the Tangent of A B, and in the Triangle ABC, it will be $\frac{x}{\sqrt{1-x^2}} : d :: 1 : \frac{a}{\sqrt{x^2-a^2}}$, therefore $\frac{ax}{\sqrt{1-x^2}} = d$ $\overline{\sqrt{x^2-a^2}}$; hence $x^4+\overline{\frac{a^2}{d^2}-a^2-1}\times x^2=-a^2$: Putting $-2b=\frac{a^2}{d^2}-a^2-1$, we have $x^4-2bx^2=-a^2$, this solved, gives $x=\sqrt{b\pm\sqrt{b^2-a^2}}=,75506556$, or to $,3982512$, the Sines of 49° 1′ 51″ and 23° 28′, the Latitude and Declination required.

Problems on the Maxima and Minima of Quantities.

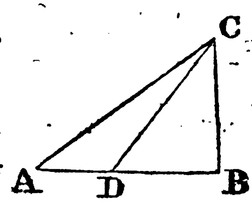
Here (as in Fluxions) the Operation for determining a Maximum is the same as that for determining a Minimum; and when a Quantity is required to be the greatest or least that the Conditions of the Problem can possibly admit of, that Quantity is called a Maximum, or Minimum; and the Moment that it becomes such, it is at a stand, and neither increases nor decreases: Therefore to calculate it by Algebra, add an exceeding small Quantity e to the unknown Number x; write $x+e$ and its Powers for x and its Powers in the Maximum or Minimum, reserving those Terms only in which the first Power of e is involved, make the Result equal to nothing; and if the Equation, after Reduction, has several Roots, they must be tried se-

we have $x^{\frac{5}{6}}=\frac{4}{9}$, therefore $x=\overline{\frac{4}{9}}\Big|^{\frac{6}{5}}$: This Equation multiplied by 3 (to augment the Number under the Radical Sign), gives $3x=\overline{\frac{4096}{2187}}\Big|^{\frac{1}{5}}$, and the Logarithm of $\frac{4096}{2187}$ is 0.2725111, which, being divided by 5, the Quotient .05450222 is the Logarithm of 1.1337108, therefore $3x=1.1337108$, and consequently $x=0.3779036$.

The Value of x might have been likewise found, by the Method demonstrated for extracting the Roots of pure Powers.

EXAMPLE IV.

In the Triangle ABC, there is given $AB=5$, and $DB=3$, to find the perpendicular BC, when the Angle ACD is a Maximum.



Put $AB=a$, $DB=c$, and $BC=x$ then by plain Trigonometry, $a : x :: 1 : \frac{x}{a} =$ the Tangent of the Angle at A, and $c : x :: 1 : \frac{x}{c} =$ the Tangent of the Angle BDC, their Difference is $\frac{\overline{a-c}\times x}{ac+x^2}$ the Tangent of the Angle ACD, which is Maximum per Question writing $x+e$ for x, we have $\frac{\overline{a-c}\times x+\overline{a-c}\times e}{ac+2ex+x^2}=\frac{\overline{a-c}\times x}{ac+x^2}$, or $\frac{x+e}{ac+2ex+x^2}=\frac{x}{ac+x^2}$; hence $x^3+ex^2+acx+ace=x^3+2ex^2+acx$, and $ex^2=ace$, or $x^2=ac$, therefore $x=\sqrt{ac}=3.8729833$.

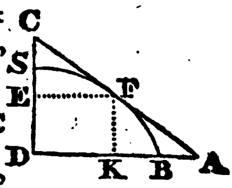
And since $x^2=ac$, therefore $a : x :: x : c$, which shews that BC is a mean Proportional between AB and DB.

In

In clearing the Equation of Fractions, the best Way is to omit multiplying and setting down those Terms in which the first Power of e will not be included. See the Sixth, Seventh, and Eighth Examples.

EXAMPLE V.

Required the Dimensions and Area of the greatest Semi-parabola that can be inscribed in a Right-Angled Triangle DAC, whose Sides CD, DA, are 12 and 16 Inches respectively.



In the annexed Figure, let $DK = EF = x$, $CD = a$, and $DA = b$, then, by similar Triangles, $b : a :: x : \frac{ax}{b} = EC$, and $\frac{ax}{2b} = ES = SC$ by the Property of the Parabola, hence $a - \frac{ax}{2b}$, or $\frac{2ab-ax}{2b} = DS$; and $\frac{ax}{2b} : x^2 :: \frac{2ab-ax}{2b} : 2bx - x^2 = \overline{DB}^2$: consequently $\overline{2bx-x^2}|^{\frac{1}{2}} \times \frac{\overline{2ab-ax}}{3b}$ is the Area of the inscribed Semi-parabola DSB, which is a Maximum per Question, therefore $\overline{2bx-x^2}|^{\frac{1}{2}} \times \overline{2b-x}$ is a Maximum, and its Square $x^4 - 6bx^3 + 12b^2x^2 - 8b^3x$ is likewise a Maximum: Here by writing $x+e$ for x, and proceeding as in the first Example, we have $4x^3e - 18bx^2e + 24b^2xe - 8b^3e = 0$, or $x^3 - 4\frac{1}{2}bx^2 + 6b^2x - 2b^3 = 0$; this Equation is divisible by $x - \frac{1}{2}b$, without any Remainder, therefore $x = \frac{1}{2}b = 8$: Hence by writing $\frac{1}{2}b$ for x, the Abscissa DS, or $\frac{2ab-ax}{2b}$ becomes $= \frac{3}{4}a = 9$, the Semi-ordinate $DB = \sqrt{\frac{3}{4}b^2} = \frac{1}{2}b\sqrt{3} = 13.8564064$; and the required Area is $\frac{1}{2}b\sqrt{3} \times \frac{\quad}{2} = \frac{1}{4}ab\sqrt{3} = 83.1384384$ Square Inches.

EXAMPLE VI.

Find the Diameter and Depth of a Cylindrical Tub with the least internal Superficies possible to contain 174 $\frac{13}{188}$ Ale Gallons.

Put $s=49087,5$ the Cubic Inches in 174 $\frac{13}{188}$ Ale Gallons, $c=3.1416$; $x=$ the Diameter, and $y=$ the Depth of the Tub; then $\frac{cx^2y}{4}=s$, hence $y=\frac{4s}{cx^2}$, and $cxy=\frac{4s}{x}$ the inside Curve Superficies, $\frac{cx^2}{4}=$ the Area of the Bottom; therefore $\frac{cx^2}{4}+\frac{4s}{x}$, or $\frac{cx^3+16s}{4x}$ is the internal Superficies, a Minimum per Question, writing $x+e$ for x, we have $\frac{cx^3+3cex^2+16s}{x+e}=\frac{cx^3+16s}{x}$, hence $3cex^3=cex^3+16es$, or $cx^3=8s$, and $x=2\sqrt[3]{\frac{s}{c}}=50$ Inches, consequently, $y=\frac{4s}{cx^2}=\frac{4s}{4c\sqrt[3]{\frac{s^2}{c^2}}}=\sqrt[3]{\frac{s}{c}}$ $=25$.

EXAMPLE VII.

The Area and Perimeter of Right-Angled Parallelogram are expressed by the same Number of Chains; and the Square of its Breadth multiplied into its Length is a Maximum: Quere its Dimensions?

Put x for the Length, and y for the Breadth of the required Parallelogram; then will its Area be $xy=2x+2y$ $=$ its

$=$ its Perimeter per Question, whence $y=\frac{2x}{x-2}$, and $xy^2=x\times\frac{4x^2}{x^2-2x+4}=\frac{4x^3}{x^2-2x+4}$ a Maximum; writing $x+e$ for x, we have $\frac{x^3+3ex^2}{x^2+2e-4\times x-4e}=\frac{x^3}{x^2-2x+4}$, and $3ex^4-12ex^3+12ex^2=2ex^4-4ex^3$, or $x^2-8x=-12$; hence $x=4+\sqrt{4}=6$ Chains, and $y=\frac{2x}{x-2}=3$.

EXAMPLE VIII.

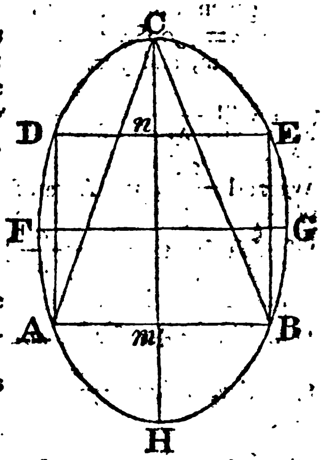
Find the internal Dimensions of a Rectangular Malt Cistern that shall be composed of the least Lead possible (at a given Thickness) to hold 25 Quarters of Barley.

Put $a=430084$ the solid Inches in 25 Quarters; $x=$ the Depth, and $y=$ the inside Length of the Cistern; then will $\frac{a}{xy}$ be its Breadth, $\frac{a}{x}$ the Area of its Bottom, $2xy$ and $\frac{2a}{y}$ the Area of its two Ends and two Sides respectively, therefore $\frac{a}{x}+2xy+\frac{2a}{y}$ is the whole internal Superficies, a Minimum per Question, in which, by writing $x+e$ for x, and looking upon y as a known Quantity, we have $\frac{a}{x+e}+2ey=\frac{a}{x}$, hence $2ex^2y=ae$, or $2x^2y=a$, therefore $x=\sqrt{\frac{a}{2y}}$: And by writing $y+e$ for y in the Minimum, (considering x as known) we have $\frac{a}{y+e}+ex=\frac{a}{y}$, and $exy^2=ae$, or $xy^2=a$, hence $y=\frac{a}{xy}$, which shews the Cistern to be Square, for $\frac{a}{xy}$ is its Breadth, as has been premised: But we have before

fore found $xy^2=a$, hence $x=\frac{a}{y^2}$, therefore $\sqrt{\frac{a}{2y}}=\frac{a}{y^2}$, or $ay^3=2a^2$, and $y=\sqrt[3]{2a}=95.10304608$ Inches, hence $x=\frac{a}{y^2}=\frac{a}{\sqrt[3]{4a^2}}=\frac{1}{2}\sqrt[3]{2a}=47.55152304$.

EXAMPLE IX.

Find the Solidity of the greatest Cone and Cylinder that can be inscribed in a Spheroid HCFG, generated by the Rotation of a Semi-ellipsis upon its transverse Diameter, which is equal to 60 Inches, and its conjugate Diameter is 40 Inches.

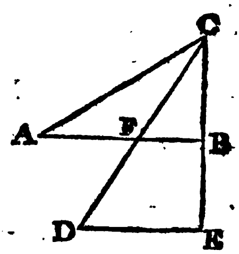


Put $HC=60=a$, $FG=40=b$, $c=3.1416$; and $Hm=Cn=x$, then will $a-x=mC$ the Altitude of the Cone ABC, and by the Property of the Ellipsis, we have $a^2 : b^2 :: \overline{a-x}\times x : \frac{b^2}{a^2}\times\overline{a-x}\times x=Am^2$, therefore $\frac{cb^2}{a^2}\times\overline{a-x}\times x$ is the Area of the Cone's Base, which multiplied by $\frac{a-x}{3}$ ($\frac{1}{3}mC$), gives $\frac{cb^2}{3a^2}\times\overline{ax-x^2}\times\overline{a-x}$ for its Solidity, which must be the greatest possible, therefore $\overline{ax-x^2}\times\overline{a-x}$, or $x^3-2ax^2+a^2x$ is a Maximum, in which by writing $x+e$ for x, we have $3ex^2-4aex+a^2e=0$, or $x^2-\frac{4}{3}ax=-\frac{1}{3}a^2$; this solved, gives $x=\frac{2}{3}a\mp\sqrt{\frac{1}{9}a^2}=\frac{2}{3}a\mp\frac{1}{3}a$, and in the present Case $x=\frac{1}{3}a=20$: Hence by writing $\frac{1}{3}a$ for x, the solid Content of the Cone becomes

$\frac{4ab^2c}{81}=14893,5111$, &c. Inches. The Solidity of the Cylinder ABDE, is $\frac{cb^2}{a^2}\times\overline{ax-x^2}\times\overline{a-2x}$, a Maximum per Question; therefore $\overline{ax-x^2}\times\overline{a-2x}$, or $2x^3-3ax^2+a^2x$ is a Maximum, in which, by writing $x+e$ for x, we have $6ex^2-6aex+a^2e=0$, or $x^2-ax=-\frac{1}{6}a^2$; hence $x=\frac{1}{2}a-\frac{a}{2\sqrt{3}}=\frac{1}{2}a-\frac{1}{6}a\sqrt{3}$, and by substituting $\frac{1}{2}a-\frac{1}{6}a\sqrt{3}$ for x in the Solidity of the Cylinder, it becomes $\frac{ab^2c\sqrt{3}}{18}=29020,85756$.

EXAMPLE X.

Two Ships are to sail at the same Time from one Port in the Latitude of 48° North, the first is to steer between the South and West, until her Departure becomes equal to 150 Geographical Miles; the other is to sail 25° 43′ more Westerly into the Latitude of 46° North: What Latitude will the first Ship be arrived in, and what Courses must these Ships steer to have the Rectangle of their Distances the least possible?



In the annexed Figure, AC and DC denote the Distances sailed.

Put $s=,4339212$ the Natural Sine to the Radius 1 of the < ACF, its Co-sine $,9009508=c$, $DE=150=a$, $BC=120=b$, and $x=$ the Sine of the < ACB, then $\sqrt{1-x^2}$ = the Sine of the < CAB; and $cx-s\sqrt{1-x^2}$ = the Sine of the < DCE: Hence, by Trigonometry, we have $cx-s\sqrt{1-x^2} : a :: 1 : \frac{a}{cx-s\sqrt{1-x^2}}=DC$, and $\sqrt{1-x^2} : b :: 1 : \frac{b}{\sqrt{1-x^2}}=AC$, therefore $AC\times DC=\frac{b}{\sqrt{1-x^2}}\times$

$\frac{a}{cx-s\sqrt{1-x^2}}=\frac{ab}{cx\sqrt{1-x^2}+sx^2-s}$ a Minimum per Question: But, the Numerator ab being a constant Quantity, it may therefore be omitted, and so by writing $x+e$ for x, in $cx\sqrt{1-x^2}+sx^2$, we have $ce\sqrt{1-x^2}-\frac{cex^2}{\sqrt{1-x^2}}$ $+2esx=0$, or $c\sqrt{1-x^2}+2sx=\frac{cx^2}{\sqrt{1-x^2}}$; hence $2cx^2-c$ $=2sx\sqrt{1-x^2}$, or $4c^2x^4-4c^2x^2+c^2=4s^2x^2-4s^2x^4$, or $4c^2x^4+4s^2x^4-4c^2x^2-4s^2x^2=-c^2$, and $x^4-x^2=\frac{-c^2}{4c^2+4s^2}$, this solved, gives $x=\sqrt{\frac{1}{2}+\sqrt{\frac{1}{4}-\frac{c^2}{4c^2+4s^2}}}=,846736$ the Sine of $57^\circ\ 52'$ the second Ship's Course, and $57^\circ\ 52$ $-25^\circ\ 43'=32^\circ\ 9'$ the first Ship's Course. As the Sine of DCE : DE :: Sine of CDE : $CE=3^\circ\ 59'$, therefore the required Latitude is $44^\circ\ 1'$ North.

FINIS.

E R R A T A.

Page 7, laſt Line, for *Denominations*, read *Denominators.*
Page 17, Line 27, for *Principle*, read *Principal.*
Page 20, Line 3, for 5*l.* 0*s.* 0*d!* read 5*l.* 0*s.* 0*d.*$\frac{1}{2}$.
Page 145, fifth Line from the Bottom, for x^2+4a^2, read $x^2=4a^2$.
Page 210, laſt Line but one, for $a^4\ ^4$, read $a^4\ x^4$.
Page 211, Line 10, for x^2 read x^4.
Page 216, Line 22, for Page 178, read 195.
Page 219, Line 3, for z^2 read z^3.
Page 224, Line 13, for Pages 107, 108, read Pages 102, 103.
Page 234, Line 4, from the Bottom, for *ſubtract* read *extract.*

Page 240, Line 14, for $x^x - \overline{x+,25}|^{\frac{\overline{x+5}|^{\frac{1}{x}}}{x}}$ read $x^x - \overline{x+,25}|^{\frac{\overline{x+,25}|^{\frac{1}{x}}}{x}}$.

Page 262, Line 19, for Page 198, read Page 201.
Page 263, Line 25, for Page 202, read Page 204.
Page 263, for $\sqrt{\overline{2x^2+2d}|^2}$ read $\overline{2x^2+2d^2}|$.

Page 304, Line 7, for $x \times \overline{1+\frac{x}{100}}| - x$, read $x \times \overline{1+\frac{x}{100}}|^x - x$ is the Intereſt.

In the Aſtronomical Draughts, commencing on Page 307, the Character of the Sun ſhould have been ☼ as in the correſpondin gOperations.

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